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Deriving weights from general pairwise comparison matrices

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Abstract

The problem of deriving weights from pairwise comparison matrices has been treated extensively in the literature. Most of the results are devoted to the case when the matrix under consideration is reciprocally symmetric (i.e., the i, j-th element of the matrix is reciprocal to its j, i-th element for each i and j). However, there are some applications of the framework when the underlying matrices are not reciprocally symmetric. In this paper we employ both statistical and axiomatic arguments to derive weights from such matrices. Both of these approaches lead to geometric mean-type approximations. Numerical comparison of the obtained geometric mean-type solutions with Saaty's eigenvector method is provided also.

Keywords: Pairwise comparison matrices; AHP; Consistency; Geometric mean

JEL classification: C43; C44

1. Introduction

The problem of deriving weights from pairwise comparisons has been studied extensively in the literature (e.g., see Barzilai et al., 1987; Chu et al., 1979; Cook and Kress, 1988; Crawford and Williams, 1985; Golany and Kress, 1993; Hartvigsen, 2005; Laslier, 1996; Saaty, 1977) and has applications in various fields (e.g., see Hovanov et al., 2004; Kerner, 1993; Laffond et al., 1996;

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Saaty, 1980; Slutzki and Volij, 2006; Troutt and Elsaid, 1996). For the purpose of this paper, it will be convenient to formulate the problem in the economic framework.

Given $n \geq 2$ infinitely divisible goods in a marketplace, an $n \times n$ positive matrix $C=(c_{ij})$ denotes the exchange rates between goods. That is, element c_{ij} (exchange-coefficient) equals the number of units of good j that can be obtained by trading one unit of good i. In the case when the matrix C under consideration is transitive (consistent), i.e.,

$$c_{ij}c_{jk} = c_{ik} \text{ for all } i,j,k = 1,\ldots,n, \tag{1}$$

there exist positive weights $w_1,\ldots, w_n$ such that

$$c_{ij} = w_i/w_j \text{ for all } i,j = 1,\ldots,n. \tag{2}$$

Otherwise the matrix is called intransitive (inconsistent). When C is transitive these weights are defined up to a positive multiplier and can be interpreted as exchange indices of the corresponding goods. The transitivity property (1) has an obvious economic interpretation — namely, it states that there is no possibility for an exchange arbitrage between different parts of the market under consideration.

The problem arising in different applications within this framework is to find weights $w_1,\ldots,w_n$ such that w_i/w_j approximate elements c_{ij} of an intransitive matrix C. This problem has been studied extensively in the literature in the case when the matrix C is *reciprocally symmetric*, i.e.,

$$c_{ij}c_{ji} = 1 \text{ for all } i, j = 1,\ldots,n. \tag{3}$$

Some of the formulations of this problem using mathematical programming (Barzilai, 1997; Crawford and Williams, 1985), axiomatic (Barzilai et al., 1987), or functional (Narasimhan, 1982) approaches have led to the geometric mean

$$w_i = \left(\prod_{j=1}^{n} c_{ij}\right)^{\frac{1}{n}}, i = 1,\ldots,n \tag{4}$$

of a corresponding row of the matrix C as the solution. However, for some applications the assumption of reciprocal symmetry may be too restrictive. For example, in the case when goods under exchange are currencies, a coefficient c_{ij} is a rate of exchange of currency i in relation to currency j. The presence of transaction costs makes the considered matrix $C=(c_{ij})$ intransitive and, in particular, Eq. (3) no longer holds.

In this paper we generalize some results for weights derived in the literature for reciprocally symmetric matrices to the case of arbitrary positive matrices satisfying $c_{ii}=1$, $i=1,\ldots,n$. To derive weights we will resort to two different approaches. The first statistical approach in Section 2 constructs an optimal (in the logarithmic least-squares sense) transitive approximation of a general positive pairwise comparison matrix. The second axiomatic approach in Section 3 develops a characterization of the functional form of weights that is motivated by its underlying mathematical properties, including ordinal invariance (Section 3.1), the presence of physical dimension (Section 3.2), the existence of a normalizing condition (Section 3.3), or the decomposability condition (Section 3.4). We find that these two different approaches lead to almost the same solutions — namely, the geometric mean-type functional that reduces to Eq. (4) in the reciprocally symmetric case. Finally, in Section 4 we compare the obtained geometric mean-type functional with the principal eigenvector method of deriving weights from a pairwise comparison matrix proposed by Saaty (1977) for the Analytic Hierarchy Process (AHP).

To begin the analysis we introduce several assumptions and notational conventions. We suppose that a pairwise comparison matrix $C=(c_{ij})$ is positive, and its diagonal elements do not contain information on $w_1,\ldots,w_n$. This is not an assumption but rather a logical necessity. For simplicity, it is supposed that all pairwise comparison matrices under consideration satisfy $c_{ii}=1$, $i=1,\ldots,n$.

We denote by $\mathbf{R}$ and $\mathbf{R}_+$ the set of real numbers and the set of positive real numbers, respectively. We use a boldface letter such as $\boldsymbol{\alpha}$ to represent the vector $(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_n)$ for given $i\in\{1,\ldots,n\}$. When a non-boldface letter like α is used for a vector, it will be taken to mean the vector $(\underbrace{\alpha,\ldots,\alpha}_{n-1})$. The terms $\mathbf{c}^i$ and $\mathbf{c}_i$ denote the i-th row $(c_{i1},\ldots,c_{ii-1},c_{ii+1},\ldots,c_{in})$ and i-th column $(c_{1i},\ldots,c_{i-1i},c_{i+1i},\ldots,c_{ni})$ of a matrix $C=(c_{ij})$, respectively. All operations with vectors will be performed component-wise, e.g., $\boldsymbol{\alpha}\mathbf{c}^i=(\alpha_1c_{i1},\ldots,\alpha_{i-1}c_{ii-1},\alpha_{i+1}c_{ii+1},\ldots,\ \alpha_nc_{in})$, $\mathbf{c}_i/\alpha=(c_{1i}/\alpha,\ldots,c_{i-1i}/\alpha,\ c_{i+1i}/\alpha,\ldots,\ c_{ni}/\alpha)$, etc. The unitary $n\times n$ matrix is denoted by $\mathbf{I}$.

We assume that the weights $w_1,\ldots,w_n$ derived from the elements c_{ij} satisfy some desirable properties. Let $C=(c_{ij})$ be a pairwise comparison matrix. We call weights $w_i\ (C)$, $i=1,\ldots,n$ of the matrix C *appropriate* if they satisfy the following conditions:

(i) $w_i(C)$, $i=1,\ldots,n$ is positive and depends on only $\mathbf{c}^i$ and $\mathbf{c}_i$; $w_i(C)$, $i=1,\ldots,n$ is strictly increasing with elements of $\mathbf{c}^i$, and decreasing with elements of $\mathbf{c}_i$;
(ii) $w_i(C)$, $i=1,\ldots,n$ satisfy Eq. (2) if matrix C is transitive; and
(iii) the weight of a good is independent under permutations of the set of goods $\{1,\ldots,n\}$.

The listed assumptions are commonly used in the literature on deriving weights from pairwise comparison matrices. The main statement of assumption (i) is that the weight w_i attributed to alternative i is independent of relative measurements among alternatives other than i (e.g., see Barzilai, 1997, Axiom 1). Assumption (ii) explains why $w_i\ (C)$ can be considered as an approximation to weight w_i from Eq. (2) and is used in all methods of deriving weights from pairwise comparison matrices (e.g., see Barzilai et al., 1987, Axiom 1; Cook and Kress, 1988; Hartvigsen, 2005, Property 1). Assumption (iii) means that a weight's value does not depend on the description of the problem (e.g., see Barzilai et al., 1987, Axiom 2).

Assumptions (i) and (iii) imply that $w_i(C)=w(\mathbf{c}^i;\ \mathbf{c}_i)$, where function w: $\mathbf{R}_+^{n-1}\times\mathbf{R}_+^{n-1}\rightarrow\mathbf{R}_+$ is symmetric for pairwise permutations $c_{ij}\leftrightarrow c_{ik}$ and $c_{ji}\leftrightarrow c_{ki}$, strictly increasing with $\mathbf{c}^i$, and decreasing with $\mathbf{c}_i$. In this paper we will deal with appropriate weights only.

2. Statistical derivation

Slightly modifying the approach of Crawford and Williams (1985), we suppose that elements c_{ij}, $i\neq j$ of a pairwise comparison matrix C can be treated as ratios $\tilde{w}_i/\tilde{w}_j$ of components of independent realizations of a random vector $(\tilde{w}_1,\ldots,\tilde{w}_n)$. We assume that the random vector $(\ln\tilde{w}_1,\ldots,\ln\tilde{w}_n)$ has multivariate normal distribution with the vector of means $(\ln w_1,\ldots,\ln w_n)$ and the diagonal covariance matrix $\sigma^2\mathbf{I}$ with some (possibly unknown) variance $\sigma^2>0$. Then c_{ij}, $i\neq j$ are independent realizations of lognormal random variables with parameters $(\ln(w_i/w_j),\ 2\sigma^2)$. We notice that the lognormal assumption is quite commonplace in the case when goods under exchange are currencies and coefficients c_{ij} are corresponding exchange rates (for a theoretical justification of the lognormal distribution of errors in the judgment of dissimilarities, see Ramsay, 1977).

Estimators for w_i can be considered as the median value of $\tilde{w}_i$, and estimated variance σ^2 can be treated as a measure of adequacy of corresponding transitive approximation of C. We restrict ourselves to maximum likelihood estimations.

The log-likelihood function of the sample is (up to a strictly increasing linear transformation)

$$L(C|w_1,\dots,w_n;\sigma^2) = -n(n-1)\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i\neq j}^{n}\left(\ln c_{ij} - \ln(w_i/w_j)\right)^2. \tag{5}$$

The form of log-likelihood function leads to logarithmic least-square (LLS) approximation for $w_1,\dots,w_n$. Simple calculus gives the following proposition.

Proposition 1. *For a given positive pairwise comparison matrix C, the maximum likelihood estimators for $w_1,\dots,w_n$ and σ^2 are given by the formulas*

$$\hat{w}_i(C) = A\left(\prod_{j=1}^{n}\frac{c_{ij}}{c_{ji}}\right)^{\frac{1}{2n}}, i=1,\dots,n, \tag{6}$$

$$\hat{\sigma}^2(C) = \frac{1}{2n(n-1)}\sum_{i\neq j}^{n}\left(\ln c_{ij} - \ln(\hat{w}_i/\hat{w}_j)\right)^2, \tag{7}$$

where A is an arbitrary positive constant.

Obviously, in the case of reciprocally symmetric matrices, estimator (6) reduces to Eq. (4) (up to a positive multiplier). Thus, we can conclude that Eq. (4) gives the weight of good i if the matrix is transitive and that it is the best (in the LLS sense) approximation if the matrix is reciprocally symmetric. It is easy to check that the resulting formula (6) generates appropriate vectors of weights (i.e., satisfying conditions (i)–(iii)).

The same approach allows us to characterize the geometric mean aggregation rule (e.g., see Barzilai and Golany, 1994) for pairwise comparisons matrices. Using the previous reasoning, we suppose that a researcher has at his disposal a collection of pairwise comparisons matrices $C^{(t)}=(c_{ij}^{(t)})$, $t=1,\dots,T$, which represent, say, states of the market at different times $t=1,\dots,T$. It is assumed that the t-th pairwise comparison matrix is induced by the normal vector $(\ln\tilde{w}_1,\dots,\ln\tilde{w}_n)$ with the vector of means $(\ln w_1,\dots,\ln w_n)$ and the diagonal covariance matrix $\sigma_t^2\,\mathbf{I}$ with known variance σ_t^2. The corresponding log-likelihood function of the sample is (up to a strictly increasing linear transformation)

$$L(C^{(1)},\dots,C^{(T)}|w_1,\dots,w_n;\sigma^2) = -n(n-1)\sum_{t=1}^{T}\ln\sigma_t^2 - \sum_{t=1}^{T}\frac{1}{2\sigma_t^2}\sum_{i\neq j}^{n}\left(\ln c_{ij}^{(t)} - \ln(w_i/w_j)\right)^2. \tag{8}$$

Simple calculus gives the following proposition.

Proposition 2. *Under the aforementioned assumptions, the maximum likelihood estimators for $w_1,\dots,w_n$ are given by the formula*

$$\hat{w}_i = A\prod_{t=1}^{T}\left(\prod_{j=1}^{n}\frac{c_{ij}^{(t)}}{c_{ji}^{(t)}}\right)^{\frac{Pt}{2n}} = A\prod_{t=1}^{T}\hat{w}_i^{Pt}(C^{(t)}), i=1,\dots,n, \tag{9}$$

where $\hat{w}_i\,(C^{(t)})$ *is an LLS-optimal approximation* (6) *of weight of good i in economic state t, A is a positive constant, and* $p_t = \frac{1/\sigma_t^2}{\sum_{\tau=1}^{T} 1/\sigma_\tau^2}$, $t = 1, \ldots, T$.

3. Axiomatic derivations

In this section we give axiomatic characterizations of the functional form of weights $w_1,\ldots,w_n$ of an intransitive pairwise comparison matrix C by enumerating its mathematical properties.

3.1. The ordinal invariance condition

The economic nature of the framework implies that the amount of any good from $\{1,\ldots,n\}$ is measured by a ratio scale (e.g., prices are stated in terms of exchange rates of value per unit of some good). In this subsection we assert preservation of the ordinal structure of weights under an admissible transformation of this scale. Following Aczél (1990), we call appropriate weights of a positive pairwise comparison matrix *ordinally invariant* if for any two pairwise comparison matrices C and C' the values

$$\operatorname{sgn}\left\{w(\boldsymbol{\alpha}\mathbf{c}^i;\mathbf{c}_i/\boldsymbol{\alpha}) - w(\boldsymbol{\alpha}\mathbf{c}'^i;\mathbf{c}'_i/\boldsymbol{\alpha})\right\},\ \ i = 1,\ldots,n \tag{10}$$

are independent of $\boldsymbol{\alpha} \in \mathbf{R}_+^{n-1}$. Turning back to the example with currencies, assumption (10) can be interpreted as follows. Let matrices C and C' describe a currency market at two different points of time (say, yesterday and today). Suppose that, applying the comparative analysis, we come to a conclusion that the exchange index of i-th currency has decreased in time ($w_i(C) \geq w_i(C')$). Of course, it is natural to expect that this conclusion does not depend on the chosen measurement scale (Eq. (10)).

In the case of reciprocally symmetric matrices, relation (10) also can be considered as a generalization of the assumption that w is a homomorphism from the set of reciprocally symmetric $n \times n$ matrices to a set of positive weights vectors (see Barzilai et al., 1987, Axiom 3).

Proposition 3. *For* $n \geq 3$, *continuous ordinally invariant appropriate weights* w_i *of a pairwise comparison matrix C have the form*

$$w_i(C) = w(\mathbf{c}^i;\mathbf{c}_i) = h(\mathbf{c}^i\mathbf{c}_i)\left(\prod_{j=1}^{n} c_{ij}\right)^{1/n},\ \ i = 1,\ldots,n, \tag{11}$$

where $h\colon \mathbf{R}_+^{n-1} \to \mathbf{R}_+$ *is some strictly decreasing continuous symmetric function such that* $h(c_1 c, c_2,\ldots,c_n)c^{1/n}$ *is increasing in* $c>0$ *for any vector* $\mathbf{c} \in \mathbf{R}_+^{n-1}$.

Proof. For a fixed $i \in \{1,\ldots,n\}$, by ordinal invariance as defined in Eq. (10), we obtain

$$w(\mathbf{c}^i;\mathbf{c}_i)\left\{\begin{matrix} > \\ = \end{matrix}\right\}w(\mathbf{c}'^i;\mathbf{c}'_i)\ \text{ if and only if }\ w(\boldsymbol{\alpha}\mathbf{c}^i;\mathbf{c}_i/\boldsymbol{\alpha})\left\{\begin{matrix} > \\ = \end{matrix}\right\}w(\boldsymbol{\alpha}\mathbf{c}'^i;\mathbf{c}'_i/\boldsymbol{\alpha}) \tag{12}$$

for any $\boldsymbol{\alpha} \in \mathbf{R}_+^{n-1}$. Following arguments of Aczél and Moszner (1994, p. 11), we deduce that Eq. (12) holds if and only if there exists an order preserving mapping $F(\cdot,\boldsymbol{\alpha})$ such that for all $\mathbf{c}^i$, $\mathbf{c}$, $\boldsymbol{\alpha} \in \mathbf{R}_+^{n-1}$

$$w(\boldsymbol{\alpha}\mathbf{c}^i;\mathbf{c}_i/\boldsymbol{\alpha}) = F\left(w(\mathbf{c}^i;\mathbf{c}_i),\boldsymbol{\alpha}\right). \tag{13}$$

Under the continuity assumption, the general form of the function F given property (iii) is found by Aczél (1990, Theorem 11):

$$F(v,\boldsymbol{\alpha}) = f^{-1}\left(f(v)\prod_{j\neq i}\alpha_j^r\right), \tag{14}$$

where f: $w(\mathbf{R}_+^{n-1};\mathbf{R}_+^{n-1})\to\mathbf{R}_+$ is some continuous strictly monotonic function, and r is a constant.

To solve function f let us consider a transitive matrix $C=(c_{ij})$. For arbitrary $k\neq i$ and given property (ii), we should have

$$c_{ik} = \frac{w(\mathbf{c}^i;\mathbf{c}_i)}{w(\mathbf{c}^k;\mathbf{c}_k)} = \frac{w(\mathbf{c}^i;1/\mathbf{c}^i)}{w(\mathbf{c}^k;1/\mathbf{c}^k)} = \frac{f^{-1}\left(f(w(1;1))\prod_{j\neq i}c_{ij}^r\right)}{f^{-1}\left(f(w(1;1))\prod_{j\neq k}c_{kj}^r\right)} = \frac{f^{-1}\left(f(w(1;1))\prod_{j\neq i}c_{ij}^r\right)}{f^{-1}\left(f(w(1;1))c_{ki}^{nr}\prod_{j\neq i}c_{ij}^r\right)}. \tag{15}$$

As $n\geq 3$, values $\prod_{j\neq i}c_{ij}^r$ and c_{ki} can be chosen independently. Upon taking $\prod_{j\neq i}c_{ij}^r = f(1)/f(w(1;1))$, we get

$$\frac{1}{c_{ki}} = c_{ik} = \frac{1}{f^{-1}(f(1)c_{ki}^{nr})}, \tag{16}$$

or, alternatively,

$$f(c) = f(1)c^{nr}. \tag{17}$$

Setting $\boldsymbol{\alpha}=1/\mathbf{c}^i$ in Eq. (13), and using Eq. (17), we get

$$w(1;\mathbf{c}^i\mathbf{c}_i) = F(w(\mathbf{c}^i;\mathbf{c}_i),1/\mathbf{c}^i) = f^{-1}\left(f(w(\mathbf{c}^i;\mathbf{c}_i))\prod_{j\neq i}c_{ij}^{-r}\right) = w(\mathbf{c}^i;\mathbf{c}_i)\prod_{j\neq i}c_{ij}^{-1/n}. \tag{18}$$

Denoting $h(\mathbf{c})=w(1;\mathbf{c})$, $\mathbf{c}\in\mathbf{R}_+^{n-1}$, we obtain Eq. (11). Given properties (i) and (iii), h is a strictly decreasing continuous symmetric function such that $h(c_1c, c_2,\ldots, c_n)c^{1/n}$ is increasing in $c>0$ for any fixed positive vector $\mathbf{c}$. □

Obviously, if matrix $C=(c_{ij})$ is reciprocally symmetric, then ordinally invariant (11) weights reduce (up to a positive multiplier) to the geometric mean as defined in Eq. (4).

Closing this subsection, we discuss the technical requirements used in Proposition 3. The assumption $n\geq 3$ is substantial. Indeed, let g be an arbitrary one-place continuous positive function satisfying $g(c)=cg(1/c)$, and h be strictly decreasing one-place continuous function such that $h(cc')g(c)$ is increasing in $c>0$ for any fixed positive c'. Then the weight-function

$$w(c;c') = h(cc')g(c) \tag{19}$$

satisfies requirements of Proposition 3 for $n=2$. In contrast, the continuity requirement can be relaxed (for details see a number of results in Aczél and Moszner, 1994).

One can easily check that the appropriateness of weights and ordinal invariance are independent.

3.2. The dimensionality condition

In this subsection we suppose that the amount (quantity, volume) of good i has some physical dimension $[u_i]$. Thus, an element of matrix C can be represented in the form $c_{ij}\,[u_j/u_i]$, where c_{ij} is the value of the exchange-coefficient and $[u_j/u_i]$ is its physical dimension. Turning back to the example with currencies, $u_1,\ldots,u_n$ symbolize currency descriptions (European euro, US dollar, etc.), e.g., c_{ij}[EUR/USD].

When the "old" measurement units u_i, u_j are converted to "new" units u_i', u_j' (related by $u_i=\alpha_i u_i'$, $u_j=\alpha_j\, u_j'$, $\alpha_i>0$, $\alpha_j>0$), the corresponding exchange-coefficient can be expressed as

$$c_{ij}'[u_j'/u_i'] = c_{ij}[\alpha_j u_j/\alpha_i u_i] = (\alpha_j/\alpha_i)c_{ij}[u_j/u_i], \tag{20}$$

which indicates the quantity c_{ij}' of j-th good's "new" units u_j' that can be exchanged for one "new" unit u_i' of good i.

Our main supposition in this subsection will be that appropriate weights have some physical dimension as a function of $u_1,\ldots, u_n$ of measurement units. This property gives such indices attributes amenable to economic interpretation. To be more precise, we call appropriate weights $w(\mathbf{c}^i;\, \mathbf{c}_i)$, $i=1,\ldots, n$ of a positive pairwise comparison matrix $C=(c_{ij})$ *dimensional* if they can be represented in the form

$$w_i(C) = w(\mathbf{c}^i[\mathbf{u}/u_i]; \mathbf{c}_i[u_i/\mathbf{u}]) = V(\mathbf{c}^i; \mathbf{c}_i)D([\mathbf{u}/u_i]; [u_i/\mathbf{u}]), \tag{21}$$

where the function V is the weight's value, and D is the measurement unit of a corresponding weight as a function of $u_1,\ldots, u_n$.

Assumptions (20) and (21) are close to well-known arguments of dimensional analysis (Bridgman, 1932). The following proposition states that these assumptions are related to those used in the previous subsection.

Proposition 4. *Dimensional appropriate weights of a pairwise comparison matrix C have the following form:*

$$w_i(C) = h(\mathbf{c}^i\mathbf{c}_i)\left(\prod_{j=1}^{n} c_{ij}\right)^{1/n}\left[\left(\prod_{j=1}^{n}\frac{u_j}{u_i}\right)^{1/n}\right], i = 1,\ldots,n, \tag{22}$$

where function h: $\mathbf{R}_+^{n-1}\to\mathbf{R}_+$ is as in Proposition 3.

Proof. From properties (i) and (iii) it follows that functions w,V,D: $\mathbf{R}_+^{2(n-1)}\to\mathbf{R}_+$ are symmetric under pairwise permutations. Let us set in representation (21) $\mathbf{c}^i=\mathbf{c}_i=1$; then we have

$$w([\mathbf{u}/u_i]; [u_i/\mathbf{u}]) = V(1; 1)D([\mathbf{u}/u_i]; [u_i/\mathbf{u}]). \tag{23}$$

Denoting $v(\mathbf{c}^i;\, \mathbf{c}_i)=V(\mathbf{c}^i;\, \mathbf{c}_i)/V(1;1)$, and using Eqs. (21) and (23), we get

$$w(\mathbf{c}^i[\mathbf{u}/u_i]; \mathbf{c}_i[u_i/\mathbf{u}]) = v(\mathbf{c}^i; \mathbf{c}_i)w([\mathbf{u}/u_i]; [u_i/\mathbf{u}]). \tag{24}$$

For a fixed $i\in\{1,\ldots,n\}$ let us change measurement units from $\mathbf{u}$ to $\mathbf{u}'=\boldsymbol{\alpha}\mathbf{u}$, with $\boldsymbol{\alpha}>0$ and $u_i'=u_i$. Iterating Eq. (24) and using Eq. (20), we obtain

$$v(\boldsymbol{\alpha}\mathbf{c}^i; \mathbf{c}_i/\boldsymbol{\alpha}) = v(\mathbf{c}^i; \mathbf{c}_i)v(\boldsymbol{\alpha}; 1/\boldsymbol{\alpha}). \tag{25}$$

By substituting $\mathbf{c}^i=1/\mathbf{c}_i=\mathbf{c}$ into Eq. (25), we obtain the $(n-1)$-dimensional power Cauchy functional equation

$$v(\boldsymbol{\alpha}\mathbf{c}; 1/\boldsymbol{\alpha}\mathbf{c}) = v(\mathbf{c}; 1/\mathbf{c})v(\boldsymbol{\alpha}; 1/\boldsymbol{\alpha}) \tag{26}$$

with respect to the function $f(\boldsymbol{\alpha})=v(\boldsymbol{\alpha};1/\boldsymbol{\alpha})$. Its general symmetric positive strictly increasing in $\boldsymbol{\alpha}$ solution is (e.g., see Aczél and Roberts, 1989):

$$v(\boldsymbol{\alpha};1/\boldsymbol{\alpha}) = \prod_{j\neq i} \boldsymbol{\alpha}_j^r, \tag{27}$$

where r is a positive constant.

Let us set $\boldsymbol{\alpha}=1/\mathbf{c}^i$ in Eq. (25), or

$$v(1;\mathbf{c}^i\mathbf{c}_i) = v(\mathbf{c}^i;\mathbf{c}_i)\prod_{j\neq i} c_{ij}^{-r}, \tag{28}$$

and denote $g(\mathbf{c})=v(1;\mathbf{c})$. Then the general strictly increasing in $\mathbf{c}^i$ and strictly decreasing in $\mathbf{c}_i$ symmetric solution $v(\mathbf{c}^i;\mathbf{c}_i)$ of the functional Eq. (25) is given by the formula

$$v(\mathbf{c}^i;\mathbf{c}_i) = g(\mathbf{c}^i\mathbf{c}_i)\prod_{j\neq i} c_{ij}^{r}, \tag{29}$$

where g: $\mathbf{R}_+^{n-1}\rightarrow\mathbf{R}_+$ is a strictly decreasing symmetric function such that $g\,(c_1c, c_2,\ldots, c_n)c^{1/n}$ is increasing in $c>0$ for arbitrary positive vector $\mathbf{c}$ and $g(1)=1$. From property (ii) it follows that $r=1/n$.

Since representation $\mathbf{c}^i[\mathbf{u}/u_i]$ is permutable, it follows from Eq. (24) that

$$w([\mathbf{u}/u_i];[u_i/\mathbf{u}]) = v([\mathbf{u}/u_i];[u_i/\mathbf{u}])w(1;1). \tag{30}$$

Thus,

$$w(\mathbf{c}^i[\mathbf{u}/u_i];\mathbf{c}_i[u_i/\mathbf{u}]) = h(\mathbf{c}^i\mathbf{c}_i)\left(\prod_{j=1}^{n} c_{ij}[u_j/u_i]\right)^{1/n}, \tag{31}$$

where $h(\mathbf{c}^i\mathbf{c}_i)=g(\mathbf{c}^i\mathbf{c}_i)w(1;1)$. □

Obviously, if matrix $C=(c_{ij})$ is reciprocally symmetric, then Eq. (22) reduces (up to a positive multiplier) to the geometric mean as defined in Eq. (4). One can easily check that the appropriateness of weights and dimensionality condition are independent.

3.3. Existence of a normalizing condition

A convenient property of the geometric mean approximation (4) and its generalization (6) is the presence of the normalizing condition:

$$\prod_{i=1}^{n} w_i(C) = \text{const.} \tag{32}$$

Barzilai (1997, Theorem 1) proved that, if normalizing condition (32) holds for any consistent matrix, then weights of a reciprocally symmetric matrix satisfying (i) and (ii) have (up to a positive multiplier) the form (4). In this subsection we show that under certain conditions the existence of any normalizing property for non-transitive matrices characterizes geometric mean-type solutions. We begin with the reciprocally symmetric case.

Proposition 5. *Let $w_i\,(C)$, $i=1,\ldots,n$ be continuously differentiable with respect to elements of C and $\frac{\partial w_i(C)}{\partial c_{jk}}\neq 0$ whenever exactly one subscript from j and k is equal to i. Suppose that there exists a differentiable function G with $G_i{}' = \frac{\partial G(w_1,\ldots,w_n)}{\partial w_i}\neq 0$, $i=1,\ldots, n$ such that*

$$G(w_1(C),\ldots,w_n(C)) = 0, \tag{33}$$

for any reciprocally symmetric matrix C. Then for sufficiently large n appropriate weights of a reciprocally symmetric matrix have the form (4) *(up to a positive multiplier).*

Proof. From (i)–(iii) it follows that for a reciprocally symmetric pairwise comparison matrix C $w_i(C)=w(\mathbf{c}^i)$ for some symmetric strictly increasing function w: $\mathbf{R}_+^{n-1}\to\mathbf{R}_+$. By reciprocity of C and (i), Eq. (33) can be rewritten as

$$g(v(\mathbf{x}^1),\ldots,v(\mathbf{x}^n))=0, \tag{34}$$

where $g(w_1,\ldots,w_n)=G(e^{w_1},\ldots,e^{w_n})$, $v(\mathbf{x}^i)=\ln w(\mathbf{c}^i)$, $X=\ln C$. Different i, j, and k are fixed from $\{1,\ldots,n\}$ (as n is sufficiently large to have enough subscripts from which to choose). Differentiating Eq. (34) with respect to $x_{ij}=-x_{ji}$, $x_{jk}=-x_{kj}$, and $x_{ki}=-x_{ik}$,we get:

$$g_j'(\cdot)v_i'(\mathbf{x}^j)=g_i'(\cdot)v_j'(\mathbf{x}^i),g_k'(\cdot)v_i'(\mathbf{x}^k)=g_i'(\cdot)v_k'(\mathbf{x}^i),g_k'(\cdot)v_j'(\mathbf{x}^k)=g_j'(\cdot)v_k'(\mathbf{x}^j), \tag{35}$$

where v_i' and g_i' are the partial derivatives of corresponding functions with respect to the ith argument. As v_i' and g_i' are non-vanishing, from Eq. (35) it follows that

$$\frac{v_k'(\mathbf{x}^i)}{v_j'(\mathbf{x}^i)}\cdot\frac{v_i'(\mathbf{x}^j)}{v_k'(\mathbf{x}^j)}=\frac{v_i'(\mathbf{x}^k)}{v_j'(\mathbf{x}^k)}. \tag{36}$$

Since each fraction has only two arguments used in other fractions of Eq. (36), there exist positive continuous functions f_{kj}, f_{ik}, and f_{ij} such that

$$\frac{v_k'(\mathbf{x}^i)}{v_j'(\mathbf{x}^i)}=f_{kj}(-x_{ki},x_{ij}),\frac{v_i'(\mathbf{x}^j)}{v_k'(\mathbf{x}^j)}=f_{ik}(-x_{ij},x_{jk}),\frac{v_i'(\mathbf{x}^k)}{v_j'(\mathbf{x}^k)}=f_{ij}(x_{ki},-x_{jk}). \tag{37}$$

By Eq. (36), functions f_{kj}, f_{ik}, and f_{ij} satisfy the Sincov-type functional equation:

$$f_{kj}(-x_{ki},x_{ij})f_{ik}(-x_{ij},x_{jk})=f_{ij}(x_{ki},-x_{jk}),x_{ki},x_{ij},x_{jk}\in\mathbf{R}. \tag{38}$$

Its continuous solution is given by (see, e.g., Aczél, 1966, §8.1.3):

$$f_{kj}(x_{ki},x_{ij})=\frac{f_k(-x_{ki})}{f_j(x_{ij})},f_{ik}(x_{ij},x_{jk})=\frac{f_i(-x_{ij})}{f_k(x_{jk})},f_{ij}(x_{ki},x_{jk})=\frac{f_i(x_{ki})}{f_j(-x_{jk})}, \tag{39}$$

where f_i, f_j, f_k are arbitrary non-vanishing continuous functions. By continuity of f_i, f_j, f_k, they are either positive or negative everywhere. Without loss of generality, they can be chosen positive. Hence

$$\frac{v_k'(\mathbf{x}^i)}{v_j'(\mathbf{x}^i)}=\frac{f_k(x_{ik})}{f_j(x_{ij})}\text{ for any }i,j,k\text{ from }\{1,\ldots,n\}. \tag{40}$$

Integrating Eq. (40) and using symmetry of v, for a given i we get:

$$v(\mathbf{x})=H\left(\sum_{l\neq i}^{n}F(x_l)\right), \tag{41}$$

where F, $F'=f>0$ and H, $H'\neq 0$ are some differentiable functions. From Eq. (36) it follows that f is symmetric:

$$f(-x)=f(x). \tag{42}$$

By (ii) H and F satisfy the functional equation:

$$H\left(\sum_{l\neq i}^{n}F(x_l)\right)=x_j+H\left(\sum_{l\neq i,j}^{n}F(x_l-x_j)+F(-x_j)\right). \tag{43}$$

Differentiating Eq. (43) with respect to x_l and x_k, l, $k\neq j$ (as n is sufficiently large to have enough subscripts from which to choose), we get

$$\frac{f(x_l)}{f(x_l-x_j)}=\frac{f(x_k)}{f(x_k-x_j)}. \tag{44}$$

Taking in Eq. (44) $x_l=x$, $x_j=-y$, $x_k=0$, we obtain the exponential Cauchy functional equation

$$f(x)f(y)=f(0)f(x+y) \tag{45}$$

with respect to the function $f(x)/f(0)$. Its general positive continuous solution, which satisfies Eq. (42), is a constant (e.g., see Aczél, 1966, p. 38). Hence, F is linear. Combining this fact with Eq. (43), we get

$$v(\mathbf{x})=\frac{1}{n}\sum_{l\neq i}^{n}x_l+b, \tag{46}$$

where b is a constant. Reverse transformations lead to Eq. (4) (up to a positive multiplier). □

Proposition 6. *Under the assumptions of Proposition 5, let Eq. (33) hold for any pairwise comparison matrix C. Then for sufficiently large n appropriate weights have the form*

$$w_i(C)=w(\mathbf{c}^i;\mathbf{c}_i)=A\left(\prod_{j=1}^{n}h(c_{ij},c_{ji}^{-1})\right)^{1/n}, i=1,\dots,n, \tag{47}$$

where $A>0$ is a constant, and $h(c, c')$ is an increasing continuously differentiable Cauchy mean of positive values c and c' satisfying

$$h(c,1/c')h(c',1/c)=1 \text{ for any } c,c'\in\mathbf{R}_+. \tag{48}$$

Proof. Differentiating Eq. (33) with respect to c_{ij} and c_{ji}, $i\neq$j, we get:

$$G_i'(\cdot)w_j'(\mathbf{c}^i;\mathbf{c}_i)+G_j'(\cdot)w_{n+i}'(\mathbf{c}^j;\mathbf{c}_j)=0, G_i'(\cdot)w_{n+j}'(\mathbf{c}^i;\mathbf{c}_i)+G_j'(\cdot)w_i'(\mathbf{c}^j;\mathbf{c}_j)=0, \tag{49}$$

where G_i' and w_i' are the partial derivative of corresponding functions with respect to the i-th argument. As G_i' and w_i' are non-vanishing, from Eq. (49) it follows that

$$\frac{w_j'(\mathbf{c}^i;\mathbf{c}_i)}{w_{n+j}'(\mathbf{c}^i;\mathbf{c}_i)}=\frac{w_{n+i}'(\mathbf{c}^j;\mathbf{c}_j)}{w_i'(\mathbf{c}^j;\mathbf{c}_j)}=f(c_{ij},c_{ji}) \tag{50}$$

for some positive two-place continuous function f. Symmetry of w and Eq. (50) imply

$$f(c_{ij}, c_{ji}) f(c_{ji}, c_{ij}) = 1. \tag{51}$$

Integrating Eq. (50), we get:

$$w(\mathbf{c}^i; \mathbf{c}_i) = H(F(c_{i1}; c_{1i}), \ldots, F(c_{in}, c_{ni})) \tag{52}$$

where F and H are some continuously differentiable functions. From (iii) it follows that H is symmetric. By Eq. (51) there exists continuously differentiable function g such that

$$F(c_{ij}, c_{ji}) = g(F(c_{ji}, c_{ij})). \tag{53}$$

As F is continuous, strictly increasing in the first variable, and strictly decreasing in the second one, without loss of generality, we may assume that

$$F(c, 1/c) = c. \tag{54}$$

Eq. (54) implies $g(c)=1/c$. Hence, weights of a matrix C are equal to weights of reciprocally symmetric matrix C' with elements $c'_{ij}=F(c_{ij}, c_{ji})$:

$$w(\mathbf{c}^i, \mathbf{c}_i) = w(F(c_{i1}, c_{1i}), \ldots, F(c_{in}, c_{ni}); F(c_{i1}, c_{1i})^{-1}, \ldots, F(c_{in}, c_{ni})^{-1}). \tag{55}$$

Using Eq. (55) and Proposition 5, we get Eq. (47) with $h(c, c')=F(c, 1/c')$. Given strict increases of h in both variables, Eq. (54) is equivalent to the internality property of h

$$\min\{c, c'\} \le h(c, c') \le \max\{c, c'\}, c, c' \in \mathbf{R}_+, \tag{56}$$

i.e., $h(c, c')$ is an increasing continuously differentiable Cauchy mean of c and c' satisfying Eq. (48) (by Eq. (53)). □

One can easily check that appropriateness of weights and the existence of the normalizing condition (33) are independent. The sufficiently large n assumption in Proposition 5 is essential. Indeed, the continuously differentiable weight-function $w(c)$, which satisfies $w(c)=cw(1/c)$, admits requirements of Proposition 5 for $n=2$. Unfortunately, we failed to prove the necessity of the continuous differentiability assumption in both Propositions.

3.4. The decomposability condition

The decomposability assumption deals with the sequence of weight estimates $w_1^{(n)}, \ldots, w_n^{(n)}$, $n=2, 3, \ldots$ and supposes that weights of a pairwise comparison matrix C do not change if any submatrix C_k of C (obtained from C by deleting the last $n-k$ rows and the last $n-k$ columns) is replaced with its transitive approximation. That is, for any $2 \le k < n$, $n \ge 3$

$$w_l^{(n)}(C) = w_l^{(n)}(C'), l = 1, \ldots, n \text{ whenever } c'_{ij} = \begin{cases} \dfrac{w_i^{(k)}(C_k)}{w_j^{(k)}(C_k)} & \text{if } \max\{i,j\} \le k \\ c_{ij} & \text{otherwise} \end{cases}. \tag{57}$$

Eq. (57) can be considered as a version of the decomposability assumption used by Kolmogorov (1930) and Nagumo (1930) for characterization of the quasi-arithmetic mean. The intuition for this condition is that the same rule for calculating weights is applicable to any subsection of the economic marketplace. Also, any exchange-coefficient can be replaced with its estimate by this rule without changing the result. In effect, any segment of the market can be considered independent of other segments of the market. Assumption (57) is quite strong and again leads to geometric mean-type weight approximations. As before, we begin with the reciprocally symmetric case.

Proposition 7. *Let the sequence of weights $w_1^{(n)},\ldots, w_n^{(n)}$, $n=2, 3,\ldots$ satisfy Eq. (57) for any reciprocally symmetric matrix C; for each i and n $w_i^{(n)}(C)$ is continuously differentiable with respect to elements of C; $\frac{\partial w_i^{(n)}(C)}{\partial c_{jk}}\neq 0$ whenever exactly one subscript from j and k is equal to i. If for each n weights $w_1^{(n)},\ldots, w_n^{(n)}$ are appropriate, then for sufficiently large n they have the form (4) (up to a positive multiplier).*

Proof. From (i)–(iii) it follows that $w_i^{(n)}(C)=w^{(n)}(\mathbf{c}^i)$ for some sequence of symmetric strictly increasing functions $w^{(n)}$: $\mathbf{R}_+^{n-1}\to\mathbf{R}_+$, $n=2, 3,\ldots$. Eq. (57) with $k=n-1$ implies

$$w^{(n)}(\mathbf{c}^{n-1}) = w^{(n)}\left(\frac{w^{(n-1)}(c_{n-11},\ldots,c_{n-1n-2})}{w^{(n-1)}(c_{12},\ldots,c_{1n-1})},\ldots,\frac{w^{(n-1)}(c_{n-11},\ldots,c_{n-1n-2})}{w^{(n-1)}(c_{n-21},\ldots,c_{n-2n-1})},c_{n-1n}\right). \quad (58)$$

Define

$$w_c(c_1,\ldots,c_{n-3}) = w^{(n-1)}(c_1,\ldots,c_{n-3},c). \quad (59)$$

Let $c_{1n-1}=\ldots=c_{n-2n-1}=c$ and c_{n-1n} be fixed, then Eq. (58) can be rewritten as

$$G(w_c(c_{12},\ldots,c_{1n-2}),\ldots,w_c(c_{n-21},\ldots,c_{n-2n-3})) = 0 \quad (60)$$

for some symmetric function G. An equation of type (60) was already considered in Proposition 5. Define $x=\ln c$, $v(\ln \mathbf{c}, \ln c)=\ln w_c(\mathbf{c})$. Then from Proposition 5 we know that there exist differentiable functions F_x, $F'_x=f_x>0$ and H_x, $H'_x\neq 0$ such that

$$v(x_1,\ldots,x_{n-3}) = H_{x_{n-3}}\left(\sum_{i=1}^{n-4}F_{x_{n-3}}(x_i)\right) \quad (61)$$

and

$$f_x(-y) = -f_x(y) \text{ for any } x \text{ and } y. \quad (62)$$

By (ii) H and F satisfy the functional equation:

$$v(x_1,\ldots,x_{n-3}) = H_{x_{n-3}}\left(\sum_{i=1}^{n-4}F_{x_{n-3}}(x_i)\right) = x_{n-3} + H_0\left(\sum_{i=1}^{n-4}F_0(x_i-x_{n-3})+F_0(-x_{n-3})\right). \quad (63)$$

From symmetry of v, it follows that

$$x_{n-3}+H_0\left(\sum_{i=1}^{n-4}F_0(x_i-x_{n-3})+F_0(-x_{n-3})\right) = x_1+H_0\left(\sum_{i=2}^{n-3}F_0(x_i-x_1)+F_0(-x_1)\right). \quad (64)$$

Differentiating Eq. (64) with respect to j and k (not equal to 1 and $n-2$, as n is sufficiently large), we get

$$\frac{f_0(x_j - x_{n-3})}{f_0(x_j - x_1)} = \frac{f_0(x_k - x_{n-3})}{f_0(x_k - x_1)}. \tag{65}$$

This equation is of type (44). Reasoning analogous to Eqs. (44)–(46) leads to

$$v(x_1, \ldots, x_{n-3}) = \frac{1}{(n-2)} \sum_{i=1}^{n-3} x_i + b \tag{66}$$

as a solution. Reverse transformations lead to Eq. (4) (up to a positive multiplier). □

Proposition 8. *Given the suppositions of Proposition 7, Eq. (57) holds for any positive pairwise comparison matrix C. Consequently, for sufficiently large n appropriate weights have the form (47).*

Proof. From (i)–(iii) it follows that $w_i^{(n)}(C) = w^{(n)}(\mathbf{c}^i; \mathbf{c}_i)$ for some sequence of functions $w^{(n)}$: $\mathbf{R}_+^{n-1} \times \mathbf{R}_+^{n-1} \to \mathbf{R}_+$, $n=2, 3, \ldots$, symmetric for pairwise permutations $c_{ij} \leftrightarrow c_{ik}$ and $c_{ji} \leftrightarrow c_{ki}$, strictly increasing with $\mathbf{c}^i$, and decreasing with $\mathbf{c}_i$. From Eq. (57) with $k=n-1$, it follows that

$$\begin{aligned} w^{(n)}(\mathbf{c}^{n-1}; \mathbf{c}_{n-1}) = w^{(n)}\Bigg(& \frac{w^{(n-1)}(c_{n-11}, \ldots, c_{n-1n-2}; c_{1n-1}, \ldots, c_{n-2n-1})}{w^{(n-1)}(c_{12}, \ldots, c_{1n-1}; c_{21}, \ldots, c_{n-11})}, \ldots, \\ & \times \frac{w^{(n-1)}(c_{n-11}, \ldots, c_{n-1n-2}; c_{1n-1}, \ldots, c_{n-2n-1})}{w^{(n-1)}(c_{n-21}, \ldots, c_{n-2n-1}; c_{1n-2}, \ldots, c_{n-1n-2})}, c_{n-1n}; \\ & \times \frac{w^{(n-1)}(c_{12}, \ldots, c_{1n-1}; c_{21}, \ldots, c_{n-11})}{w^{(n-1)}(c_{n-11}, \ldots, c_{n-1n-2}; c_{1n-1}, \ldots, c_{n-2n-1})}, \ldots, \\ & \times \frac{w^{(n-1)}(c_{n-21}, \ldots, c_{n-2n-1}; c_{1n-2}, \ldots, c_{n-1n-2})}{w^{(n-1)}(c_{n-11}, \ldots, c_{n-1n-2}; c_{1n-1}, \ldots, c_{n-2n-1})}, c_{nn-1}\Bigg). \end{aligned} \tag{67}$$

Define

$$w_{c,c'}(c_1, \ldots, c_{n-3}; c'_1, \ldots, c'_{n-3}) = w^{(n-1)}(c_1, \ldots, c_{n-3}, c; c'_1, \ldots, c'_{n-3}, c'). \tag{68}$$

Let $c_{1n-1} = \ldots = c_{n-2n-1} = c$, $c_{n-11} = \ldots = c_{n-1n-2} = c'$ and c_{n-1n}, c_{nn-1} be fixed, then Eq. (67) can be rewritten as

$$G(w_{c,c'}(c_{12}, \ldots, c_{1n-2}; c_{21}, \ldots, c_{n-21}), \ldots, w_{c,c'}(c_{n-21}, \ldots, c_{n-2n-3}; c_{1n-2}, \ldots, c_{n-3n-2})) = 0 \tag{69}$$

for some symmetric function G. Eq. (69) is already considered in Proposition 6. From Proposition 6 (Eq. (55)) we know that

$$\begin{aligned} w_{c,c'}(c_1, \ldots, c_{n-3}; c'_1, \ldots, c'_{n-3}) = w_{c,c'}(& h(c_1, c_1'^{-1}), \ldots, h(c_{n-3}, c_{n-3}'^{-1}); \\ & h(c_1, c_1'^{-1})^{-1}, \ldots, h(c_{n-3}, c_{n-3}'^{-1})^{-1}), \end{aligned} \tag{70}$$

where $h(c, c')$ is an increasing differentiable Cauchy mean of c and c', which satisfies Eq. (48). From relation (70) and the symmetry of $w^{(n-1)}$ under pairwise permutations (iii), it follows that $w_{c,c'}$ actually depends only on $h(c, c'^{-1})$. So we can apply Proposition 7 to get Eq. (47). □

One can easily check that the appropriateness of weights and the existence of the decomposability assumption (57) are independent. The sufficiently large n assumption in Proposition 7 is essential. Indeed, the decomposability assumption (57) introduces no additional information for reciprocally symmetric matrices with $n=3$. Unfortunately, we failed to prove the necessity of continuous differentiability assumption in both Propositions.

4. A comment on numerical comparison with Saaty's eigenvector method

In this section we compare the obtained geometric mean-type functional (6) with probably the best-known method of deriving weights from a pairwise comparison matrix — namely, the principal eigenvector method proposed by Thomas Saaty (1977) for AHP. In spite of the fact that the Saaty's method was originally applied to reciprocally symmetric matrices, it can be applied also to the case when

$$c_{ij}c_{ji} \geq 1 \text{ for all } i, j = 1, \dots, n. \tag{71}$$

Indeed, minor alterations of Theorem 1 of Saaty (1977) and its related Corollary yield for a pairwise comparison matrix C that satisfies inequalities (71):

$$\lambda_{\max} \geq n, \tag{72}$$

where $\lambda_{\max}$ is the principal eigenvalue of C. Moreover, the equality in relation (72) is attained if and only if C is consistent. If inequalities (71) do not hold, then $\lambda_{\max}=n$ generally does not imply consistency of C.

To compare these methods we consider a multiplicative variation $C_\varepsilon=(c_{ij}e^{\varepsilon ij})$ of a consistent matrix $C=(c_{ij})$, where $\varepsilon_{ij}=O(\varepsilon)$, $i,j=1,\dots, n$ ($\varepsilon_{ii}=0$, $i=1,\dots, n$), ε — an infinitesimal. We denote by $w_k^{(E)}(C)$ and $w_k^{(G)}(C)$, respectively, the k-th component of the eigenvector of C and the k-th weight generated by the geometric mean-type functional (6). Using classical results on perturbation of the principal eigenvalue $\lambda_{\max}$ of a positive matrix (Stewart, 1973, p. 305; Harker, 1987), we get

$$\frac{\partial \ln w_k^{(E)}(C)}{\partial \ln c_{ij}} = \begin{cases} \dfrac{n-1}{n^2}, & \text{if } i = k \\ -\dfrac{1}{n^2}, & \text{otherwise} \end{cases}, i \neq j. \tag{73}$$

Analogously,

$$\frac{\partial \ln w_k^{(G)}(C)}{\partial \ln c_{ij}} = \begin{cases} (2n)^{-1}, & \text{if } i = k \\ -(2n)^{-1}, & \text{if } j = k, i \neq j \\ 0, & \text{otherwise} \end{cases}. \tag{74}$$

Thus,

$$\ln \frac{w_k^{(E)}(C_\varepsilon)}{w_k^{(G)}(C_\varepsilon)} = \frac{\sum_{j=1}^{n} (\varepsilon_{kj} + \varepsilon_{jk})}{2n} - \frac{\sum_{i,j} \varepsilon_{ij}}{n^2} + o(\varepsilon), i \neq j. \tag{75}$$

In particular, for reciprocally symmetric perturbations ($\varepsilon_{ij}=-\varepsilon_{ji}$)

$$\ln \frac{w_k^{(E)}(C_\varepsilon)}{w_k^{(G)}(C_\varepsilon)} = o(\varepsilon), k = 1, \dots, n. \tag{76}$$

Comparing Eqs. (75) and (76), we conclude that these two methods are close only for reciprocally symmetric perturbations.

5. Conclusion

Some applications of the pairwise comparison framework deal with matrices that are not reciprocally symmetric. In this paper we employ both statistical (Proposition 1) and axiomatic arguments (Propositions 3–8) to derive weights from such pairwise comparison matrices. Both of these approaches lead to geometric mean-type approximations that reduce to the geometric mean of a corresponding row for reciprocally symmetric pairwise comparison matrices. One can easily obtain weighted versions of Propositions 3–8 by dropping the symmetry axiom (iii). The derived geometric mean-type functional is numerically compared with the principal eigenvector method proposed by Saaty. It is shown that the weights computed by the two methods for reciprocally asymmetric matrices are quite different.

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