Mikhail V. Sokolov

INTERVAL SCALABILITY OF RANK-DEPENDENT UTILITY

ABSTRACT. Luce and Narens (1985) showed that rank-dependent utility (RDU) is the most general interval scale utility model for binary lotteries. It can be easily established that this result cannot be generalized to lotteries with more than two outcomes. This paper suggests several additional conditions to ensure RDU as the only utility model with the desired property of interval scalability in the general case. The related axiomatizations of some special cases of RDU of independent interest (the quantile utility, expected utility, and Yaari's dual expected utility) are also given.

KEY WORDS: rank-dependent utility, interval scalability, meaningfulness.
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1. Introduction

The normative approach to the theory of choice under uncertainty suggests to write down a suitable set of axioms from which the model could be derived. Axioms can be conditionally divided into two parts, "contextual" (determined by the problem under investigation) and "context-free" (independent of the problem settings). Axioms of the last group are appealing in the sense that they could be applied to any problem. One possible source for "context-free" axioms is the representational theory of measurement (Krantz et al., 1971; Luce et al., 1990; Narens, 2002). This argues to focus on those algorithms of data analysis that lead to conclusions that are stable with respect to a change of a measurement scale of the input variables. In the literature this postulate is known as the requirement of meaningfulness (Luce et al., 1990, chapter 22; Narens, 2002; Roberts, 1979). The concept of meaningfulness is often formalized in terms of invariance with respect to some transformations (usually, admissible transformations of a measurement scale). Informally, one shall say that a statement involving scales is meaningful if its truth or falsity is unchanged when admissible transformations are applied to all of the scales in the statement (Roberts, 1979, p. 59). The requirement of meaningfulness is intuitive and is topical in connection with the fact that the choice of a particular measurement scale is subjective.

Various types of meaningfulness and invariance conditions are used to characterize utility functionals and risk attitudes (e.g., see Abbas, 2010, Bell and Fishburn, 2000, Ovchinnikov, 2002, Pfanzagl, 1959, Rothblum, 1975, Quiggin and Chambers, 2004, to mention just a few). This paper develops the approach of Luce and Narens (1985, section 7) and Luce (1988) to provide an axiomatization of rank-dependent utility (RDU) (Quiggin, 1982; Schmeidler, 1989) on the basis of the property of meaningfulness and stochastic dominance. Luce and Narens (1985, section 7) showed that the dual bilinear utility is the most general interval scale utility model for binary lotteries. In the usual probabilistic framework this reduces to RDU. In other words, RDU is the only cardinal utility model (in the sense that utility is defined up to an affine transformation) in the binary case. This result cannot be generalized to lotteries with more than two outcomes. Following the idea of Luce (1988) this paper suggests possible additional conditions to ensure RDU as the only cardinal utility model in the general case. Axiomatizations of some special cases of RDU of independent interest (the quantile utility, expected utility, and Yaari's dual expected utility) are also given.

The paper is organized as follows. Section 2 presents the basic definitions and notation used in the paper. In section 3 meaningfulness and stochastic dominance are used to axiomatize RDU. Motivation and economic interpretations of these assumptions are given. Subsection 3.1 collects some auxiliary results on preference relations that satisfy these two conditions. Subsection 3.2 shows that these assumptions are necessary and sufficient for RDU in the case of binary lotteries. This case is dealt with in a related manner in Luce and Narens (1985, section 7). Section 3.3 argues that these conditions are not determinative for RDU in the general case. This generates the need for an additional assumption. Decomposability, (restricted) branch independence, and ordinal bisymmetry are suggested as such an assumption. Finally, section 4 presents the related axiomatizations of some special cases of RDU of independent interest: the quantile utility (subsection 4.1), Yaari's dual expected utility (subsection 4.2), and expected utility (subsection 4.3). All proofs are given in the Appendix.

2. Preliminaries

By R and R_{++} denote the set of real numbers and the set of positive real numbers, respectively. All operations with vectors and sets are performed component-wise, e.g. $T(\boldsymbol{x}) = \left(T(x_1),...,T(x_n)\right)$, where T is a function of one variable, $\boldsymbol{x} = (x_1,...,x_n)$. The $(n-1)$-dimensional unit simplex $\left\{\boldsymbol{p} = (p_1,..., p_n) \geq 0 : \sum_{i=1}^{n} p_i = 1\right\}$ is denoted by P^n.

Let X be an open real interval. A family $\mathrm{T(X)}$ of increasing bijections of X onto itself is called a *scale group* (or a *group of admissible transformations*) if it forms a group with respect to the functional composition operation $\circ$. The neutral element of $\mathrm{T}(\mathrm{X})$, the identity transformation, is denoted by id. A scale group is said to be 2-*point homogeneous* (Narens, 2002, p. 54) if

for any $x_1 < x_2$ and $x_1' < x_2'$ in X there exists $T \in \mathrm{T}(\mathrm{X})$ such that $T(x_i) = x_i'$, $i = 1, 2$. (1)

A scale group is 2-*point unique* (Narens, 2002, p. 54) if $T \in \mathrm{T}(\mathrm{X})$ appearing in (1) is unique for particular x_i, x_i', $i = 1, 2$.

Well-known examples of 2-point homogeneous scale groups are (e.g., see Narens, 2002, p. 54) the group

$$\mathrm{T}_u(\mathrm{X}) = \{T : T(x) = u^{-1}(au(x) + b), a \in \mathrm{R}_{++}, b \in \mathrm{R}, x \in \mathrm{X}\}, \quad (2)$$

where u is an increasing bijection of X onto R, describing an *interval scale* when u is affine and scales conjugate to an interval scale for a general u (Narens, 2002, p. 52); the automorphism group

$$\mathrm{T}_{\mathrm{A}}(\mathrm{X}) = \{T : T \text{ is an increasing bijection of } \mathrm{X} \text{ onto itself}\}, \quad (3)$$

describing an *ordinal scale*. Note: interval scales are 2-point unique; ordinal scales are not 2-point unique.

For a given integer $N \geq 2$ let $\mathrm{D}_{\mathrm{X}}^{(N)}$ be the set of all cumulative distribution functions, concentrated on at most N points in an open interval X:

$$\mathrm{D}_{\mathrm{X}}^{(N)} = \left\{F_{(\boldsymbol{x};\boldsymbol{p})}(x) = \sum_{i=1}^{n} 1_{[x_i,\infty)}(x) p_i,\ \boldsymbol{x} \in \mathrm{X}^n, x_1 \leq \ldots \leq x_n, \boldsymbol{p} \in \mathrm{P}^n, n \in \{1, \ldots, N\}\right\},$$

where 1_A is the indicator function of a set A. The set of all probability distributions F over the real line such that $\mathrm{supp} F$ (the support of F) is discrete and $\inf(\mathrm{supp} F), \sup(\mathrm{supp} F) \in \mathrm{X}$ will be denoted by $\mathrm{D}_{\mathrm{X}}^{(\infty)}$. An element of the set $\mathrm{D}_{\mathrm{X}}^{(N)}$ is called a (*ranked*) *lottery with* n *outcomes* and for convenience is denoted by $(\boldsymbol{x}; \boldsymbol{p})$. For the degenerate lottery $(x;1)$ at the point $x \in \mathrm{X}$ we use the special notation δ_x. The k-th partial sum of probabilities of a lottery $(\boldsymbol{x}; \boldsymbol{p})$ is denoted by $p^{(0)} = 0$; $p^{(i)} = p^{(i-1)} + p_i$, $i = 1, \ldots, n$.

By a *preference relation* ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$), we mean a complete and transitive binary relation on $\mathrm{D}_{\mathrm{X}}^{(N)}$ (weak order), with ($\mathrm{D}_{\mathrm{X}}^{(N)}, \sim$), ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succ$) defined as usual. By the above, ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) satisfies the *coalescing* property (e.g., see Luce, 2000, section 5.3.2). That is for any $n \in \{2, \ldots, N\}$, $i \in \{1, \ldots, n-1\}$, $\boldsymbol{x}$ with $x_i = x_{i+1} = x$, and $\boldsymbol{p}$, the lottery $(\boldsymbol{x}; \boldsymbol{p})$ with n outcomes is indifferent to the lottery $(x_1, \ldots, x_{i-1}, x, x_{i+2}, \ldots, x_n; p_1, \ldots, p_{i-1}, p_i + p_{i+1}, p_{i+2}, \ldots, p_n)$ with $n-1$ outcomes, these are two different ways of writing the same probability distribution. In particular, this allows us not to distinguish the degenerate lotteries $\delta_x = (x, \ldots, x; \boldsymbol{p})$ for different n and $\boldsymbol{p}$. Similarly, a lottery $(\boldsymbol{x}; \boldsymbol{p})$ with n outcomes with $p_i = 0$ for some $i \in \{1, \ldots, n\}$ is indifferent to the lottery

$\left(x_1,...,x_{i-1},x_{i+1},...,x_n;\,p_1,...,p_{i-1},p_{i+1},...,p_n\right)$ with $n-1$ outcomes. Hereafter this property will be also referred to as coalescing.

The following definitions are common in utility theory. ($D_X^{(N)}, \succeq$) is *nondegenerate* if there exist $(\boldsymbol{x};\boldsymbol{p}),(\boldsymbol{x}';\boldsymbol{p}')\in D_X^{(N)}$ such that $(\boldsymbol{x};\boldsymbol{p})\succ(\boldsymbol{x}';\boldsymbol{p}')$. A degenerate lottery δ_x is a *certainty equivalent* of $(\boldsymbol{x};\boldsymbol{p})$ if $(\boldsymbol{x};\boldsymbol{p})\sim\delta_x$. ($D_X^{(N)}, \succeq$) satisfies *stochastic dominance* if $(\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p}')$ whenever $F_{(\boldsymbol{x};\boldsymbol{p})}(x)\le F_{(\boldsymbol{x}';\boldsymbol{p}')}(x)$ for all $x\in X$. ($D_X^{(N)}, \succeq$) satisfies *strict stochastic dominance* if $(\boldsymbol{x};\boldsymbol{p})\succ(\boldsymbol{x}';\boldsymbol{p}')$ whenever $F_{(\boldsymbol{x};\boldsymbol{p})}(x)\le F_{(\boldsymbol{x}';\boldsymbol{p}')}(x)$ for all $x\in X$ with strict inequality for some x. *Simple-continuity* (Wakker, 1994) holds if for any lottery $(\boldsymbol{x};\boldsymbol{p})\in D_X^{(N)}$ with N outcomes the sets $\{\boldsymbol{x}'\in X^N : x_1'\le...\le x_N', (\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p})\}$ and $\{\boldsymbol{x}'\in X^N : x_1'\le...\le x_N', (\boldsymbol{x}';\boldsymbol{p})\succeq(\boldsymbol{x};\boldsymbol{p})\}$ are closed in X^N (note that this continuity condition concerns only variation in outcomes). ($D_X^{(N)}, \succeq$) is said to be *continuous with respect to the topology of weak convergence* if for any pair of weakly convergent sequences of probability distributions $\left(\boldsymbol{x}^{(k)};\boldsymbol{p}^{(k)}\right)\xrightarrow[k\to\infty]{}(\boldsymbol{x};\boldsymbol{p})$ and $\left(\boldsymbol{x}'^{(k)};\boldsymbol{p}'^{(k)}\right)\xrightarrow[k\to\infty]{}(\boldsymbol{x}';\boldsymbol{p}')$ $\left(\boldsymbol{x}^{(k)};\boldsymbol{p}^{(k)}\right)\succeq\left(\boldsymbol{x}'^{(k)};\boldsymbol{p}'^{(k)}\right)$ for all $k=1,2,...$ implies $(\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p}')$. ($D_X^{(N)}, \succeq$) is *representable* if there exists a *utility functional* $U: D_X^{(N)}\to R$ such that $(\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p}')$ if and only if $U(\boldsymbol{x};\boldsymbol{p})\ge U(\boldsymbol{x}';\boldsymbol{p}')$. A utility functional U is called *idempotent* if $U(\delta_x)=x$ for all $x\in X$.

($D_X^{(N)}, \succeq$) is said to be *meaningful with respect to a scale group* $T(X)$ (for short, $T(X)$-*meaningful*) if

$$(\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p}') \Rightarrow (T(\boldsymbol{x});\boldsymbol{p})\succeq(T(\boldsymbol{x}');\boldsymbol{p}') \text{ for all } T\in T(X). \tag{4}$$

The given definition asserts preservation of the preference structure under transformations $T\in T(X)$ and is equivalent to 1-meaningfulness (Narens, 2002, section 2.6) of the statement $(\boldsymbol{x};\boldsymbol{p})\succeq(\boldsymbol{x}';\boldsymbol{p}')$.

3. Rank-dependent utility

Let ($D_X^{(N)}, \succeq$) be a preference relation. *Rank-dependent utility* (RDU) (Quiggin, 1982; Schmeidler, 1989) holds if there exist a strictly increasing *utility function* $u: X\to R$ and a nondecreasing *transformation function* $w:[0,1]\to[0,1]$ with $w(0)=0$ and $w(1)=1$ such that the utility functional $U: D_X^{(N)}\to R$ defined by

$$U(\boldsymbol{x};\boldsymbol{p})=\sum_{i=1}^{n}\left[w(p^{(i)})-w(p^{(i-1)})\right]u(x_i),\ (\boldsymbol{x};\boldsymbol{p})\in D_X^{(N)} \tag{5}$$

represents $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$. If $w([0,1]) = \{0,1\}$ then the utility function u is unique up to an order preserving transformation, otherwise u is cardinal (i.e. it is defined up to a strictly increasing affine transformation). The transformation function w is uniquely determined.

Axiomatizations of RDU are obtained by Abdellaoui (2002), Chateauneuf (1999), Luce (2000), Quiggin (1993), Quiggin and Wakker (1994), Segal (1989, 1993), Wakker (1994). The dual expected utility, the important special case of RDU, is axiomatizated by Yaari (1987). Further axiomatizations of rank-dependent utility have been provided in Köbberling and Wakker (2003) and Zank (2010). In this section we give an axiomatization of (5) in the special case when $u(\mathrm{X}) = \mathrm{R}$. The axiomatization develops the approach of Luce and Narens (1985, section 7) to describe preferences over lotteries with more than two outcomes. The assumption of meaningfulness of $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ with respect to a 2-point homogeneous measurement scale and stochastic dominance play a crucial role in this axiomatization. Interpretations of these assumptions are as follows.

We give three possible interpretations of the assumption of meaningfulness with respect to a 2-point homogeneous scale group. The first one (see Luce and Narens, 1985, Luce 1988) postulates the existence of a cardinal utility, which is measured on an interval or weaker scale (a scale that preserves less structure). Indeed, if a representable preference relation is meaningful with respect to a 2-point homogeneous scale group, then its utility functional has the form $U(u(\boldsymbol{x}); \boldsymbol{p})$, where the function $u : \mathrm{X} \to \mathrm{R}$ (referred to as a cardinal utility function) is defined at least up to a positive affine transformation (see Proposition 3 below for the details). In other words, for any $a \in \mathrm{R}_{++}$, $b \in \mathrm{R}$ utility functionals $U(u(\boldsymbol{x}); \boldsymbol{p})$ and $U(au(\boldsymbol{x}) + b; \boldsymbol{p})$ induce the same preference relation on $\mathrm{D}_{\mathrm{X}}^{(N)}$; consequently the class of equivalent utility functions forms an interval scale or a weaker scale. Note that (5) satisfies this property whenever $u(\mathrm{X}) = \mathrm{R}$.

The second interpretation of meaningfulness with respect to a 2-point homogeneous scale group is the so called "size" argument (e.g., see Aczél and Moszner, 1994): this is a generalization of the condition that constant absolute risk aversion and constant relative risk aversion are both satisfied. Indeed, these two types of risk aversion hold if and only if the preference relation is meaningful with respect to the scale group $\mathrm{T}_{id}(\mathrm{R})$ (note that Yaari's dual expected utility satisfies this condition). To demonstrate the applicability of meaningfulness with respect to some other 2-point homogeneous scale groups consider a preference relation on the set of investment projects that return a fixed amount of money in a random duration. If an investor is guided by the present value criterion, then, equivalently, preferences over lotteries of the form $(xe^{-dt}; \boldsymbol{p})$ should be analyzed, where x is the project's return, d is the discount rate, and the project duration is t_i with probability p_i, $i = 1,...,n$. If the discount rate is hard to predict and project scale does not affect the investor's preferences, then the following invariance conditions make sense: preferences are independent of the discount rate d and changing the project scale $x \mapsto bx$, $b \in \mathrm{R}_{++}$. Alternatively, independence of preferences with respect to the discount rate can be interpreted as compliance with the relation of stochastic dominance of infinite order (see Corollary 2 to Theorem 4 of Fishburn, 1980) or enlarging project durations by the same factor: $\boldsymbol{t} \mapsto a\boldsymbol{t}$, $a \in \mathrm{R}_{++}$. Clearly, these conditions hold if and only if the preference relation is meaningful with respect to the scale group $\mathrm{T}_{\ln}(\mathrm{R}_{++})$ (describing a *log-interval scale*).

The third interpretation of meaningfulness with respect to a scale group $\mathrm{T}(\mathrm{X})$ is usual and sometimes referred to as the "scale" argument (e.g., see Aczél and Moszner, 1994): outcomes in the set X are assumed to be measured on a scale with the scale group $\mathrm{T}(\mathrm{X})$. Then $\mathrm{T}(\mathrm{X})$-meaningfulness asserts preservation of the preference structure under all acceptable scales. For example, meaningfulness with respect to the scale group $\mathrm{T}_{id}(\mathrm{R})$ makes sense when outcomes are dates, meaningfulness with respect to the scale group $\mathrm{T}_{\ln}(\mathrm{R}_{++})$ is reasonable when outcomes are of psychophysical nature (for example, loudness of a sound), meaningfulness with respect to the automorphism group $\mathrm{T}_{\mathrm{A}}(\mathrm{X})$ makes sense when outcomes are measured on an ordinal scale (e.g. progress in studies), etc.

The intuitive meaning of 2-point homogeneity of a scale group is a restriction on its minimal "diversity" ("dimensionality").

Stochastic dominance is a common assumption in almost all utility models. This is a probabilistic counterpart to the traditional "more is better" implication.

The implicit property of coalescing is a consequence of the fact that $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ is defined on the set of probability distributions, rather than on a set of random variables (generating these distributions on a corresponding sample space). This condition restricts attention to preferences that are state-independent (compare with Yaari, 1987, Axiom A1). However, it should be noted that the empirical evidence (e.g., see Birnbaum, 1999; Tversky and Kahneman, 1986) points to violation of stochastic dominance and coalescing.

The mentioned condition $u(\mathrm{X}) = \mathrm{R}$ is essential to the requirement of meaningfulness be well defined, implies continuity of u and can be justified by the following "transitivity" reason. For any $(\boldsymbol{x}; \boldsymbol{p}), (\boldsymbol{x}'; \boldsymbol{p}') \in \mathrm{D}_{\mathrm{X}}^{(N)}$ there exists a lottery $(\boldsymbol{x}''; \boldsymbol{p}') \in \mathrm{D}_{\mathrm{X}}^{(N)}$ such that $x_i' \neq x_i''$ at most for one $i \in \{1, \dots, n\}$ and

$$(\boldsymbol{x}; \boldsymbol{p}) \sim (\boldsymbol{x}''; \boldsymbol{p}'). \tag{6}$$

It is easy to establish that if RDU holds and there exists $p^* \in (0,1)$ such that $w(p^*) \in (0,1)$, then (6) implies $u(\mathrm{X}) = \mathrm{R}$ (if $w([0,1]) = \{0,1\}$ then u can be chosen such that $u(\mathrm{X}) = \mathrm{R}$). This condition is a version of the Archimedean axiom: any two lotteries can be equalized by changing a branch.

Finally, instead of a continuity condition, in most of our axiomatizations representability of $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ is explicitly assumed. This assumption is closely related to the certainty-equivalent condition requiring the existence for every lottery of a certainty equivalent (e.g., see Wakker, 1994). Indeed, under nondegeneracy and meaningfulness with respect to a 2-point homogeneous scale group these two assumptions are equivalent and imply the existence of a unique certainty equivalent for each lottery (see Proposition 1 below for the details). Obviously, this requirement is weaker than the continuity conditions usually used in the literature.

Compared with one of the most general axiomatizations of RDU that was provided in Abdellaoui (2002) the present axiomatization seems more restrictive: this requires richness of the outcome space (it is a continuum) and $u(\mathrm{X}) = \mathrm{R}$. These assumptions cannot be weakened and are necessary to the requirement of meaningfulness be well defined. But we find this acceptable since the task of the paper is to ascertain the conditions under which RDU is the most general form of a cardinal utility model.

3.1. Auxiliary results

In this subsection we collect some auxiliary results on preference relations that satisfy stochastic dominance and are meaningful with respect to a 2-point homogeneous scale group.

The first proposition states that representability and meaningfulness imply the existence of an idempotent utility functional.

Proposition 1.

Let $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ be a nondegenerate representable preference relation. If stochastic dominance and meaningfulness with respect to a 2-point homogeneous scale group hold, then there exists an idempotent utility functional that represents $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$.

The next proposition is a variant of a well-known result in the theory of invariant means (Orlov, 1979, p. 98; Ovchinnikov, 1996, Theorem 2.2) and deduces the functional equation for an idempotent utility functional of a meaningful preference relation.

Proposition 2.

Let an idempotent utility functional U represent ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$). ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) is $\mathrm{T}(\mathrm{X})$-meaningful if and only if U satisfies the functional equation

$$U(T(\boldsymbol{x}); \boldsymbol{p}) = T \circ U(\boldsymbol{x}; \boldsymbol{p}) \text{ for all } (\boldsymbol{x}; \boldsymbol{p}) \in \mathrm{D}_{\mathrm{X}}^{(N)} \text{ and } T \in \mathrm{T}(\mathrm{X}). \tag{7}$$

The last proposition of this subsection is based on the fundamental result of Alper and Narens (Luce et al., 1990, Theorem 5, p. 120) and specializes equation (7) in the case of 2-point homogeneous scale group.

Proposition 3.

Let ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) be a nondegenerate representable preference relation. If stochastic dominance and meaningfulness with respect to a 2-point homogeneous scale group $\mathrm{T}(\mathrm{X})$ hold, then there exists an idempotent utility functional U that represents ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$). U is continuous with respect to outcomes (i.e. for each fixed $\boldsymbol{p} \in \mathrm{P}^N$ $U(\cdot; \boldsymbol{p})$ is a continuous function) and:

- If $\mathrm{T}(\mathrm{X})$ is 2-point unique (i.e. for any $x_1 < x_2$ and $x_1' < x_2'$ in X there exists a unique $T \in \mathrm{T}(\mathrm{X})$ such that $T(x_i) = x_i'$ for $i = 1, 2$), then there exists an increasing bijection u of X onto R such that the idempotent utility functional $G(\boldsymbol{y}; \boldsymbol{p}) = u \circ U(u^{-1}(\boldsymbol{y}); \boldsymbol{p}) = u \circ U(\boldsymbol{x}; \boldsymbol{p})$, $\boldsymbol{y} = u(\boldsymbol{x})$ represents the preference relation ($\mathrm{D}_{\mathrm{R}}^{(N)}, \succeq'$) induced by the rule

$$(\boldsymbol{y}; \boldsymbol{p}) \succeq' (\boldsymbol{y}'; \boldsymbol{p}') \text{ if and only if } (u^{-1}(\boldsymbol{y}); \boldsymbol{p}) \succeq (u^{-1}(\boldsymbol{y}'); \boldsymbol{p}') \tag{8}$$

and satisfies the functional equation

$$G(a\boldsymbol{y} + b; \boldsymbol{p}) = aG(\boldsymbol{y}; \boldsymbol{p}) + b \text{ for all } (\boldsymbol{y}; \boldsymbol{p}) \in \mathrm{D}_{\mathrm{R}}^{(N)},\ a \in \mathrm{R}_{++},\ b \in \mathrm{R}; \tag{9}$$

- If $\mathrm{T}(\mathrm{X})$ is not 2-point unique, then RDU holds with $u = id$ and the transformation function of the form

$$w(p) = 1_{[c,1]}(p) \text{ or } w(p) = 1_{(c',1]}(p) \tag{10}$$

for some constants $c \in (0,1]$, $c' \in [0,1)$.

Proposition 3 reduces the problem of characterization of meaningful preference relations to solving functional equation (9). We begin with axiomatization of binary RDU ($N = 2$). This case is dealt with in a related manner in Luce and Narens (1985, section 7).

3.2. The binary case

The current theoretical literature on utility theory mainly agrees that preferences over binary lotteries can be represented by RDU. As noted by Marley and Luce (2002, p. 41), the binary case is of special interest as, constructing a utility theory, some authors work with different inductive principles beginning with binary lotteries.

Theorem 1.

Let $(D_X^{(2)}, \succeq)$ be a nondegenerate representable preference relation. Then, RDU holds with $u(X) = R$ if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to a 2-point homogeneous scale group.

It can be easily checked that (i), (ii), nondegeneracy, and representability of $(D_X^{(2)}, \succeq)$ are essential for the "if" part of Theorem 1; the requirement $u(X) = R$ is essential for the "only if" part.

3.3. The case $N > 2$

The binary case is not revealing for RDU as many other versions of utility theory (e.g., rank weighted utility) reduce to RDU for binary lotteries. It may happen that in subsection 3.2 we have characterized something more than RDU. Actually, an attempt of a direct generalization of Theorem 1 to preference relations on a set of lotteries with more than two outcomes doesn't characterize RDU and leads to quite a general construction. This is caused by the existence of a large class of solutions of functional equation (9) for $N > 2$ besides linear ones (Aczél et al., 1994, section 4). This generates the need for an additional assumption for RDU to be the only cardinal utility model. In this subsection we consider several conditions that can achieve this purpose.

Differentiability. Differentiability of a utility functional with respect to outcomes may serve as such a condition. This follows from a result of Aczél et al. (1994, Proposition 9), who showed that a differentiable solution of (9) is linear with respect to $\boldsymbol{y}$. The assumption of differentiability can be justified by simplification of optimization technique with the preference relation. Differentiability with respect to outcomes (together with differentiability with respect to probabilities) also simplifies the measure representation of a utility functional (see Green and Jullien, 1988, p. 359–360). Unfortunately, differentiability is difficult to validate and empirically impossible to test. That is why we omit the details.

Decomposability. The so called decomposability assumption usually says that any lottery with $n \geq 3$ outcomes should be indifferent to a lottery having just two outcomes, the first of which is itself a lottery composed of the $n-1$ outcomes other than the most preferred (or its the certainty equivalent) and the second of which is the most preferred one. Various versions of this assumption are considered by Chew and Epstein (1989a), Liu (2004), Luce (1988, 2000), Luce and Fishburn (1995).

We use the following variant of the decomposability assumption. A preference relation ($D_X^{(N)}, \succeq$) is called *decomposable* if for any $(\boldsymbol{x}; \boldsymbol{p}) \in D_X^{(N)}$ there exists $x \in X$ depending only on $x_1, ..., x_{n-1}$ and $\boldsymbol{p}$ such that

$$(\boldsymbol{x}; \boldsymbol{p}) \sim (x, x_n; p^{(n-1)}, 1 - p^{(n-1)}). \tag{11}$$

The value x in (11) can be interpreted as a certainty equivalent of the conditional lottery (sublottery) of $(\boldsymbol{x}; \boldsymbol{p})$ given that x_n does not realize. The only distinction of this definition from those usually used in the literature is that we make no assumption about the value of such a conditional lottery, except its independence of x_n. The intuition for decomposability is that a sublottery can be evaluated on the assumption that it is the whole lottery.

Under decomposability and coalescing, each lottery has a certainty equivalent. Indeed,

$$(\boldsymbol{x}; \boldsymbol{p}) \sim (\boldsymbol{x}, x_n; \boldsymbol{p}, 0) \sim (x, x_n; 1, 0) \sim \delta_x . \tag{12}$$

Clearly, RDU satisfies (11) with

$$x = u^{-1}\left(\sum_{i=1}^{n-1}\left[\frac{w(p^{(i)})}{w(p^{(n-1)})} - \frac{w(p^{(i-1)})}{w(p^{(n-1)})}\right]u(x_i)\right),\ w(p^{(n-1)}) \neq 0 . \tag{13}$$

Theorem 2.

Let ($D_X^{(N)}, \succeq$) be a nondegenerate preference relation. Then, RDU holds with $u(X) = R$ if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to a 2-point homogeneous scale group;

(iii) decomposability.

Note that under assumptions (i)–(iii) the value x in (11) also has RDU representation (13) with the transformation function of the form $p \mapsto w(p)/w(p^{(n-1)})$. Values

$$p(i|n-1) = w(p^{(i)})/w(p^{(n-1)}),\ i = 1, ..., n-2,\ w(p^{(n-1)}) \neq 0 \tag{14}$$

in (13) can be interpreted as the transformed conditional probabilities of $(\boldsymbol{x}; \boldsymbol{p})$ given that x_n does not realize. This interpretation can be justified by the following version of the choice property (Luce, 2000, p. 78): $p(i|j)p(j|k) = p(i|k)$ for all $1 \le i \le j \le k \le n$ and $\boldsymbol{p} \in P^n$.

Branch independence. A related axiomatization of RDU may be derived under the so called branch independence assumption (also referred to as ordinal independence, coordinate independence, or the comonotonic sure-thing principle) (e.g., see Birnbaum, 1999; Chateauneuf, 1999; Green and Jullien, 1988; Wakker, 1994).

A preference relation ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) is called *branch independent* if

$$\left(x_1,...,x_{n-1},x_n;\boldsymbol{p}\right) \succeq \left(x_1',...,x_{n-1}',x_n;\boldsymbol{p}\right) \Rightarrow \left(x_1,...,x_{n-1},x_n';\boldsymbol{p}\right) \succeq \left(x_1',...,x_{n-1}',x_n';\boldsymbol{p}\right) \tag{15}$$

for any $x_n' \geq \max\{x_{n-1}, x_{n-1}'\}$ in X.

Branch independence is a weak form of Savage's independence axiom. It states that given a preference between lotteries, common branches (the same outcome with the same probability) of the lotteries have no effect on the ordering.

Theorem 3.

Let ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) be a nondegenerate representable preference relation. Then, RDU holds with $u(\mathrm{X}) = \mathrm{R}$ if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to a 2-point homogeneous scale group;

(iii) branch independence.

Ordinal bisymmetry. An unexpected source of additional assumptions can be found in the field of information fusion and aggregation operators: if RDU holds then for a fixed $\boldsymbol{p}$ the function $G(\cdot;\boldsymbol{p})$ (defined in Proposition 3) is nothing more than an ordered weighted averaging operator (OWA) (Yager, 1988) and the corresponding idempotent utility functional $u^{-1} \circ U(\cdot;\boldsymbol{p})$ is a quasi-OWA operator (Fodor et al., 1995). There exist a number of characterization results for these operators (see Chew and Epstein, 1989b; Fodor et al., 1995; Marichal, 1998, section 4.2.5; Marichal and Mathonet, 1999); some of them may have behavioral interpretation. The next axiomatization is motivated by a result of Marichal and Mathonet (1999) and is based on the bisymmetry-like assumption.

Let each lottery in ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) has a certainty equivalent. *Ordinal bisymmetry* holds if for any $n, n' \in \{1,...,N\}$, probability vectors $\boldsymbol{p} \in \mathrm{P}^n$, $\boldsymbol{p}' \in \mathrm{P}^{n'}$, and lotteries $\left(\boldsymbol{x}^{(1)};\boldsymbol{p}\right),...,\left(\boldsymbol{x}^{(n')};\boldsymbol{p}\right) \in \mathrm{D}_{\mathrm{X}}^{(N)}$ such that $\left(\boldsymbol{x}^{(j+1)};\boldsymbol{p}\right)$ stochastically dominates $\left(\boldsymbol{x}^{(j)};\boldsymbol{p}\right)$, $j = 1,...,n'-1$, the following condition is satisfied:

$$\left(x^{(1)},...,x^{(n')};\boldsymbol{p}'\right) \sim \left(x_{(1)},...,x_{(n)};\boldsymbol{p}\right), \tag{16}$$

where $x^{(j)}$ is a certainty equivalent of the lottery $\left(\boldsymbol{x}^{(j)};\boldsymbol{p}\right)$, $j = 1,...,n'$ and $x_{(i)}$ is a certainty equivalent of the lottery $\left(x_i^{(1)},...,x_i^{(n')};\boldsymbol{p}'\right)$, $i = 1,...,n$.

To motivate ordinal bisymmetry assume that outcomes of a lottery $\left(\boldsymbol{x}^{(j)};\boldsymbol{p}\right)$ depend on the state j of nature, where states $j = 1,...,n'$ are ranked in order of less to more favorable and occur with probabilities $\boldsymbol{p}'$. To evaluate the situation a decision maker can either aggregate branches of each of the lotteries, and then aggregate lotteries by states, or aggregate states of each of the branches, and then aggregate these global branches. Ordinal bisymmetry asserts that the decision maker can choose either the first or the second manner to proceed; the result will be the same. Alternative motivations and the related assumptions are considered by Luce (2000, section 3.7.3) (rank-dependent bisymmetry), Nakamura (1992) (multisymmetry), Pfanzagl (1959) (bisymmetry), Quiggin (1982) (independence axiom).

Theorem 4.

Let $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ be a nondegenerate representable preference relation. Then, RDU holds with $u(\mathrm{X}) = \mathrm{R}$ if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to a 2-point homogeneous scale group;

(iii) ordinal bisymmetry.

Theorem 4 can be obtained as a consequence of a result of Marichal and Mathonet (1999) on characterization of OWA. An independent proof of the theorem follows from Proposition 3. We also note that instead of conditions (i) and (ii), in Theorem 4 it can be explicitly stated that RDU holds for binary lotteries. Moreover, under *strict* stochastic dominance and simple-continuity the assumption of meaningfulness in Theorem 4 can be omitted:

Theorem 5.

Let $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ be a preference relation. Then, RDU holds with a continuous utility function and a strictly increasing transformation function if and only if the following conditions are satisfied:

(i) strict stochastic dominance;

(ii) simple-continuity;

(iii) ordinal bisymmetry.

4. Selected special cases

In this final section we consider some special cases of RDU representation that are interesting in its own right: Yaari's dual expected utility, the quantile utility, and expected utility.

The following two particular cases of 2-point homogeneous scale groups are of important practical interest: $\mathrm{T}_{\mathrm{A}}(\mathrm{X})$ and $\mathrm{T}_{id}(\mathrm{R})$. They correspond to ordinal and interval scales. In the nearest two subsections we observe that meaningfulness with respect to these scale groups characterizes, respectively, the quantile utility and Yaari's dual expected utility. In the last subsection we axiomatize expected utility by meaningfulness arguments.

4.1. The quantile utility

Quantile-based decision rules (e.g., see Manski, 1988; Rostek, 2010) are an important subclass of utility models by two key characteristics of quantiles, robustness and ordinality. In this subsection we characterize them using meaningfulness arguments.

The *lower* (*upper*) *quantile functional* $Q_c : \mathrm{D}_{\mathrm{X}}^{(N)} \to \mathrm{X}$ ($Q'_{c'} : \mathrm{D}_{\mathrm{X}}^{(N)} \to \mathrm{X}$) *of order* $c \in (0,1]$ ($c' \in [0,1)$) of a distribution $F_{(\boldsymbol{x};\boldsymbol{p})}$ is defined by $Q_c(\boldsymbol{x};\boldsymbol{p}) = \inf\{x \in \mathrm{X} : F_{(\boldsymbol{x};\boldsymbol{p})}(x) \geq c\}$ ($Q'_{c'}(\boldsymbol{x};\boldsymbol{p}) = \inf\{x \in \mathrm{X} : F_{(\boldsymbol{x};\boldsymbol{p})}(x) > c'\}$).

Let $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ be a preference relation. The *quantile utility* (QU) holds if either Q_c or $Q'_{c'}$ represents $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ for some constants $c \in (0,1]$, $c' \in [0,1)$. Preferences induced by the functional Q_c ($Q'_{c'}$) are the special case of RDU with the weight-function $w(p) = 1_{[c,1]}(p)$ ($w(p) = 1_{(c',1]}(p)$) regardless of the utility function u. The following theorem applies meaningfulness arguments to characterize QU.

Theorem 6.

Let $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ be a nondegenerate representable preference relation. Then, QU holds if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to the automorphism group $\mathrm{T_A}(\mathrm{X})$.

The proof of the theorem follows directly from the second alternative of Proposition 3 (since the group $\mathrm{T_A}(\mathrm{X})$ is 2-point homogeneous, but not 2-point unique) and is omitted. Theorem 6 can also be derived from a recent result of Chambers (2009, Theorem 1 and footnote 1).

Theorem 6 has a number of possible intuitive meanings. First, this states that QU is the only reasonable utility model when outcomes are measured on an ordinal scale. Second, QU is the most general form of utility model when utility of outcomes are measured on an ordinal scale; thus, QU is the only "ordinal utility" model (compare with Manski, 1988). Third, meaningfulness with respect to the group $\mathrm{T_A}(\mathrm{X})$ may be also interpreted as independence of preferences with respect to the choice of cardinal utility function. Hence, QU is the only possible socially acceptable utility (i.e. agreed with individual preferences).

4.2. Yaari's dual expected utility

The special case of RDU with the identity utility function corresponds to the dual expected utility (DEU) of Yaari (1987). Thus, any of the obtained axiomatizations of RDU (Theorems 1–5) with $\mathrm{X} = \mathrm{R}$ characterizes DEU when meaningfulness with respect to a 2-point homogeneous scale group (axiom (ii)) is replaced by meaningfulness with respect to the group $\mathrm{T}_{id}(\mathrm{R})$. Using the mentioned interpretations of meaningfulness (see section 3), this result has two possible intuitive meanings: DEU is a reasonable utility model when either outcomes are measured on an interval scale, or constant absolute risk aversion and constant relative risk aversion are both satisfied.

4.3. Expected utility

In the previous sections, we deal with a preference relation ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) on the set $\mathrm{D}_{\mathrm{X}}^{(N)}$ of ranked lotteries; ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) can be also interpreted as a relation on the set of sequences of ordered branches $(x_1, p_1; ...; x_n, p_n)$ that are consistent with coalescing. Such a relation can be naturally extended to the set of *unranked* lotteries $\mathrm{D}_{\mathrm{X}}^{\prime(N)} = \{(x_{\sigma(1)},..., x_{\sigma(n)}; p_{\sigma(1)},..., p_{\sigma(n)}) : (\boldsymbol{x}; \boldsymbol{p}) \in \mathrm{D}_{\mathrm{X}}^{(N)}, \sigma \in \Omega_n\}$, where Ω_n is the set of permutations of the set $\{1,..., n\}$, by the rule: $(x_{\sigma(1)},..., x_{\sigma(n)}; p_{\sigma(1)},..., p_{\sigma(n)}) \sim (\boldsymbol{x}; \boldsymbol{p})$, $(\boldsymbol{x}; \boldsymbol{p}) \in \mathrm{D}_{\mathrm{X}}^{(N)}$, $\sigma \in \Omega_n$. In particular, if the preference relation is representable, then this extension induces symmetry of a corresponding utility functional $U(\boldsymbol{x}; \boldsymbol{p}) = U(x_{\sigma(1)},..., x_{\sigma(n)}; p_{\sigma(1)},..., p_{\sigma(n)})$, $\sigma \in \Omega_n$.

The intuitive meaning of the unranked form is invariance of lotteries under permutation of outcomes. It is clear that such an extension itself does not apply any restriction on ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$). But this imposes some limitations whenever it is assumed that any of the considered assumptions (such as decomposability (11), branch independence (15), or ordinal bisymmetry (16)) holds for unranked lotteries $\mathrm{D}_{\mathrm{X}}^{\prime(N)}$ rather than ranked ones $\mathrm{D}_{\mathrm{X}}^{(N)}$. For example, as it is well known (e.g., see Aczél, 1966, p. 287), being applied to unranked lotteries $\mathrm{D}_{\mathrm{X}}^{\prime(N)}$, the bisymmetry assumption (16) in Theorem 5 provides expected utility representation

$$U(\boldsymbol{x}; \boldsymbol{p}) = \sum_{i=1}^{n} p_i u(x_i) . \tag{17}$$

Although the introduced unranked (symmetric) case is more restrictive, it is also of some interest (e.g., see Krantz et al., 1971, chapter 8).

Let ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$) be a preference relation. *Expected utility* (EU) holds if there exists a strictly increasing *utility function* $u : \mathrm{X} \to \mathrm{R}$ such that the utility functional (17) represents ($\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq$).

In this subsection we give a simple axiomatization of EU (as the special case of RDU with $w(p)=p$) on the set of unranked lotteries $\mathrm{D}'^{(N)}_{\mathrm{X}}$. First, we recall that if RDU representation (5) is invariant under permutations $\sigma\in\Omega_n$ then this reduces to EU representation (17).

Proposition 4 (Quiggin and Wakker, 1994, p. 492)**.**

Let $U:\mathrm{D}'^{(N)}_{\mathrm{X}}\to\mathrm{R}$ represent $(\mathrm{D}'^{(N)}_{\mathrm{X}},\succeq)$, $N>2$. If $U\left(x_{\sigma(1)},...,x_{\sigma(n)};p_{\sigma(1)},...,p_{\sigma(n)}\right)$ has the same RDU representation for any $\sigma\in\Omega_n$, then EU holds.

Thus, under the condition of Proposition 4, any axiomatization of RDU in subsection 3.3 characterizes EU representation.

Joint receipt (**concatenation**). A characterization of EU representation on the set of unranked lotteries $\mathrm{D}'^{(\infty)}_{\mathrm{X}}$ can be obtained assuming the existence of a so called concatenation (or joint receipt) operation. Various versions of the concept of joint receipt are considered by Luce (2000), Luce and Fishburn (1995), Luce and Marley (2005). We shall give the following definition.

A *concatenation* operation $\oplus:\mathrm{D}'^{(\infty)}_{\mathrm{X}}\times\mathrm{D}'^{(\infty)}_{\mathrm{X}}\to\mathrm{D}'^{(\infty)}_{\mathrm{X}}$ is said to be defined on $\mathrm{D}'^{(\infty)}_{\mathrm{X}}$, if there exists a nonconstant symmetric function $g:\mathrm{X}^2\to\mathrm{X}$ such that g is nondecreasing with respect to the both arguments and

$$(\boldsymbol{x};\boldsymbol{p})\oplus(\boldsymbol{x}';\boldsymbol{p}')=\left(g(x_1,x_1'),g(x_1,x_2'),...,g(x_n,x_{n'}');p_1p_1',p_1p_2',...,p_np_{n'}'\right). \tag{18}$$

An intuitive meaning of (18) is receiving two independent lotteries $(\boldsymbol{x};\boldsymbol{p})$ and $(\boldsymbol{x}';\boldsymbol{p}')$ at once. Indeed, let X and X' be independent random variables defined on the same probability space with probability distributions $(\boldsymbol{x};\boldsymbol{p})$ and $(\boldsymbol{x}';\boldsymbol{p}')$, respectively. Then (18) can be interpreted as the probability distribution of the concatenated random variable $g(X,X')$. Often it is additionally assumed that the set X and the binary operation g constitute a group. For example, when outcomes denote money, it is sometimes supposed that $g(x,x')=x+x'$; in this case $\oplus$ is the operation of convolution.

Theorem 7.

Let $(\mathrm{D}'^{(\infty)}_{\mathrm{X}},\succeq)$ be a nondegenerate preference relation. Then, EU holds with $u(\mathrm{X})=\mathrm{R}$ if and only if the following conditions are satisfied:

(i) stochastic dominance;

(ii) meaningfulness with respect to a 2-point homogeneous scale group;

(iii) continuity with respect to the topology of weak convergence;

(iv) a concatenation operation $\oplus$ is defined on $\mathrm{D}'^{(\infty)}_{\mathrm{X}}$ such that if $(\boldsymbol{x}_1;\boldsymbol{p}_1)\succeq(\boldsymbol{x}_1';\boldsymbol{p}_1')$ and $(\boldsymbol{x}_2;\boldsymbol{p}_2)\succeq(\boldsymbol{x}_2';\boldsymbol{p}_2')$, then $(\boldsymbol{x}_1;\boldsymbol{p}_1)\oplus(\boldsymbol{x}_2;\boldsymbol{p}_2)\succeq(\boldsymbol{x}_1';\boldsymbol{p}_1')\oplus(\boldsymbol{x}_2';\boldsymbol{p}_2')$.

An interpretation of (iv) is that the preference ordering is compatible with the concatenation operation $\oplus$, in the sense that order between a pair of lotteries is preserved under concatenation another ordered pair of lotteries.

Concluding we notice the necessity of continuity with respect to the topology of weak convergence in Theorem 7. Indeed, the preference relation induced by the utility functional

$$U(\boldsymbol{x};\boldsymbol{p})=\min_{i:p_i>0} x_i$$

satisfies the remaining conditions of the theorem with $g(x,x')=\min\{x,x'\}$.

5. Conclusion

This paper develops the approach of Luce and Narens (1985) and Luce (1988) to provide an axiomatization of rank-dependent utility on the basis of the property of meaningfulness with respect to a 2-point homogeneous scale (interval scalability), stochastic dominance, decomposability, branch independence, and ordinal bisymmetry. Motivation and economic interpretations of these assumptions are given. Related axiomatizations of the quantile utility, Yaari's dual utility, and expected utility (as the special cases of RDU of independent interest) are also obtained. The established results may serve as an additional argument to use these utility theories.

Appendix: Proofs

Proof of Proposition 1.

Let $U:\mathrm{D}_{\mathrm{X}}^{(N)}\to\mathrm{R}$ represent ($\mathrm{D}_{\mathrm{X}}^{(N)},\succeq$) and let $\mathrm{T}(\mathrm{X})$ be the 2-point homogeneous scale group under consideration. By definition, put $u(x)=U(x,...,x;\boldsymbol{p})$. By coalescing, u is well defined (i.e. $u(x)$ is independent of n and $\boldsymbol{p}$). From meaningfulness with respect to a 2-point homogeneous scale group, stochastic dominance, and nondegeneracy of ($\mathrm{D}_{\mathrm{X}}^{(N)},\succeq$) it follows that u is strictly increasing.

We shall prove that for any $(\boldsymbol{x};\boldsymbol{p})\in\mathrm{D}_{\mathrm{X}}^{(N)}$ there exists $x\in\mathrm{X}$ such that $U(\boldsymbol{x};\boldsymbol{p})=u(x)$. Assume the converse. Suppose there exists a lottery $(\boldsymbol{x};\boldsymbol{p})$ such that $U(\boldsymbol{x};\boldsymbol{p})\neq u(x)$ for all $x\in\mathrm{X}$. Then there exists a unique $x'\in[x_1,x_n]$ such that $u(x'-\varepsilon)<U(\boldsymbol{x};\boldsymbol{p})<u(x'+\varepsilon)$ for any $\varepsilon>0$.

Obviously, u has a jump discontinuity at the point x' (if it is not, then tending ε to zero, we obtain a contradiction: $U(\boldsymbol{x};\boldsymbol{p})=u(x')$).

Given $x''\in \mathrm{X}$, by 2-point homogeneity of $\mathrm{T}(\mathrm{X})$, there exists $T\in\mathrm{T}(\mathrm{X})$ such that $T(x')=x''$. By meaningfulness, we have $U(T(\boldsymbol{x});\boldsymbol{p})\neq u\circ T(x)$ for all $x\in\mathrm{X}$ and $u\circ T(x'-\varepsilon)<U(T(\boldsymbol{x});\boldsymbol{p})<u\circ T(x'+\varepsilon)$ for any $\varepsilon>0$. Hence, $u\circ T$ has a jump discontinuity at the point x'. Since T is continuous, u is discontinuous at the point $T(x')=x''$. By the arbitrariness of the choice of x'', u is discontinuous at each point of the interval X. But a monotone function has at most a countable set of discontinuity points. This contradiction proves the proposition. ■

Proof of Proposition 2.

Let a preference relation ($\mathrm{D}_{\mathrm{X}}^{(N)},\succeq$) be $\mathrm{T}(\mathrm{X})$-meaningful and let δ_x be a certainty equivalent of $(\boldsymbol{x};\boldsymbol{p})$. Applying (4) to the relation $(\boldsymbol{x};\boldsymbol{p})\sim\delta_x$, we get

$$U(T(\boldsymbol{x});\boldsymbol{p})=U(\delta_{T(x)})=T(x)=T\circ U(\boldsymbol{x};\boldsymbol{p}),\ T\in\mathrm{T}(\mathrm{X}),$$

where the first equality follows from $\mathrm{T}(\mathrm{X})$-meaningfulness, while the second one follows from idempotence of U.

Conversely, let (7) hold. Since $T\in\mathrm{T}(\mathrm{X})$ is order preserving, if $U(\boldsymbol{x};\boldsymbol{p})\geq U(\boldsymbol{x}';\boldsymbol{p}')$, then

$U(T(\boldsymbol{x});\boldsymbol{p})=T\circ U(\boldsymbol{x};\boldsymbol{p})\geq T\circ U(\boldsymbol{x}';\boldsymbol{p}')=U(T(\boldsymbol{x}');\boldsymbol{p}')$. ■

Proof of Proposition 3.

There exists an idempotent utility functional $U:\mathrm{D}_{\mathrm{X}}^{(N)}\to\mathrm{R}$ that represents ($\mathrm{D}_{\mathrm{X}}^{(N)},\succeq$) (Proposition 1). If the scale group $\mathrm{T}(\mathrm{X})$ is 2-point unique, then the Alper–Narens theorem (Luce et al., 1990, Theorem 5, p. 120) implies the existence of an increasing bijection u of X onto R such that $\mathrm{T}(\mathrm{X})=\mathrm{T}_u(\mathrm{X})$, where $\mathrm{T}_u(\mathrm{X})$ is defined in (2). For this u put

$$G(\boldsymbol{y};\boldsymbol{p})=u\circ U(u^{-1}(\boldsymbol{y});\boldsymbol{p})=u\circ U(\boldsymbol{x};\boldsymbol{p}),\ \boldsymbol{y}=u(\boldsymbol{x}). \tag{19}$$

In the new notation equation (7) in Proposition 2 takes the form (9). G represents the preference relation ($\mathrm{D}_{\mathrm{R}}^{(N)},\succeq'$) on the set $\mathrm{D}_{u(\mathrm{X})}^{(N)}=\mathrm{D}_{\mathrm{R}}^{(N)}$ induced by the rule (8) and is idempotent. Clearly, $G(\cdot;\boldsymbol{p})$ is nondecreasing. A nondecreasing solution of equation (9) is continuous with respect $\boldsymbol{y}$ (Aczél et al., 1994, p. 447). Hence, the first alternative of Proposition 3 holds.

If the group $\mathrm{T}(\mathrm{X})$ is not 2-point unique then there exist $x'<x''<x'''$ in X and $T',T''\in\mathrm{T}(\mathrm{X})$ such that the equality $T'(x)=T''(x)$ holds exactly for two $x\in\{x',x'',x'''\}$. Since $\mathrm{T}(\mathrm{X})$ is a group, the function $T=T'^{-1}\circ T''$ is an element of $\mathrm{T}(\mathrm{X})$. We claim that

$$U(x_1,x_2;p,1-p)\in\{x_1,x_2\} \tag{20}$$

holds for all binary lotteries $(x_1, x_2; p, 1-p) \in \mathrm{D}_{\mathrm{X}}^{(2)}$.

Indeed, since X is an open interval and T defined above is continuous, then at least one of the following three possibilities holds:

- there exist $x_1 < x_2$ such that x_1, x_2 are fixed points of T and $T(x) \neq x$ for all $x \in (x_1, x_2)$.

The application of (7) with this T yields:

$$\begin{aligned} &U(x_1, x_2; p, 1-p) = U(T(x_1), T(x_2); p, 1-p) = T \circ U(x_1, x_2; p, 1-p) = \ldots \\ &\ldots = \lim_{k \to \infty} T^{(k)} \circ U(x_1, x_2; p, 1-p) \in \{x_1, x_2\}, \end{aligned} \tag{21}$$

where $T^{(k)}$ is the k-th iteration of T. In the last implication of (21) we use continuity and increase of T.

- there exist $x_1 < x_2$ such that $T(x) = x$ for all $x \in [x_1, x_2]$ and $T(x) > x$ for all $x < x_1$.[1]

From meaningfulness and stochastic dominance, it follows that for fixed x_2 and p the function $U(\cdot, x_2; p, 1-p)$ maps an interval to an interval and is nondecreasing. Therefore, it is continuous. Let $U(x_1, x_2; p, 1-p) \neq x_2$ then, by meaningfulness, there exists $x_0 \le x_1$ such that $U(x_0, x_2; p, 1-p) = x_1$ and

$$\begin{aligned} &x_1 = T(x_1) = T \circ U(x_0, x_2; p, 1-p) = U(T(x_0), T(x_2); p, 1-p) = U(T(x_0), x_2; p, 1-p) = \ldots \\ &\ldots = \lim_{k \to \infty} U(T^{(k)}(x_0), x_2; p, 1-p) = U(x_1, x_2; p, 1-p). \end{aligned} \tag{22}$$

In the last equality in (22) we use continuity and increase of T and continuity of $U(\cdot, x_2; p, 1-p)$.

- there exist $x_1 < x_2$ such that $T(x) = x$ for all $x \in [x_1, x_2]$ and $T(x) < x$ for all $x > x_2$.[1]

In the same way, we have that $U(x_1, x_2; p, 1-p) = x_2$ whenever $U(x_1, x_2; p, 1-p) \neq x_1$.

Thus in all the three cases (20) holds for some $x_1 < x_2$. By 2-point homogeneity of the scale group, if $U(x_1, x_2; p, 1-p) = x_i$ holds for some $x_1 < x_2$ and $i = \{1, 2\}$ then this holds for all $x_1 < x_2$ with the same i. By stochastic dominance, for given $x_1 < x_2$ the function $p \mapsto U(x_1, x_2; p, 1-p)$ is nonincreasing. Therefore, either $U(x_1, x_2; p, 1-p) = x_2 + 1_{[c,1]}(p)(x_1 - x_2)$ or $U(x_1, x_2; p, 1-p) = x_2 + 1_{(c',1]}(p)(x_1 - x_2)$ for some constants $c \in (0,1]$, $c' \in [0,1)$.

Thus, for a given $(\boldsymbol{x}; \boldsymbol{p}) \in \mathrm{D}_{\mathrm{X}}^{(N)}$ with $\boldsymbol{p} > 0$ there exists a unique $k \in \{1, \ldots, n\}$ such that $x_k = U(x_1, x_k; p^{(k-1)}, 1 - p^{(k-1)}) \le U(\boldsymbol{x}; \boldsymbol{p}) \le U(x_k, x_n; p^{(k)}, 1 - p^{(k)}) = x_k$.

Using coalescing, this result can be extended to lotteries with zero probabilities for some outcomes. Thus defined utility functionals can be represented in the form (5) with the identity utility function and the transformation function $w(p) = 1 - U(0,1; p, 1-p)$, i.e. the second alternative of Proposition 3 holds. Obviously, these utility functionals are continuous with respect to outcomes. ■

Proof of Theorem 1.

Necessity: The preference relation induced by the utility functional (5) with $u(\mathrm{X})=\mathrm{R}$ satisfies (i) and (ii) with the scale group $\mathrm{T}_u(\mathrm{X})$.

Sufficiency follows from Proposition 3. Its second alternative yields (5) with the transformation functions of the form (10) and an arbitrary strictly increasing utility function u. u can be chosen such that $u(\mathrm{X})=\mathrm{R}$. Hence, we need to consider only the first alternative of Proposition 3.

In the binary case equation (9) reduces to

$$G(ay_1+b,ay_2+b;p,1-p)=aG(y_1,y_2;p,1-p)+b\,,\ (y_1,y_2;p,1-p)\in\mathrm{D}_{\mathrm{R}}^{(2)}\,,\ a\in\mathrm{R}_{++}\,,\ b\in\mathrm{R}\,. \qquad (23)$$

If $y_1<y_2$, then, substituting a for $1/(y_2-y_1)$ and b for $-y_1/(y_2-y_1)$ in (23), we get

$$G(y_1,y_2;p,1-p)=(1-G(0,1;p,1-p))y_1+G(0,1;p,1-p)y_2\,. \qquad (24)$$

(24) holds also in the case $y_1=y_2$. Because of stochastic dominance, $0\le G(1,0;p,1-p)\le 1$ and the function $p\mapsto G(0,1;p,1-p)$ is nonincreasing. By coalescing, $G(0,1;0,1)=1$ and $G(0,1;1,0)=0$.

Applying the reverse transformations, we get

$$U(x_1,x_2;p,1-p)=u^{-1}((1-G(0,1;p,1-p))u(x_1)+G(0,1;p,1-p)u(x_2))\,. \qquad (25)$$

A utility functional is defined up to an order-preserving transformation, hence (25) is equivalent to (5) with $w(p)=1-G(0,1;p,1-p)$. ■

Proof of Theorem 2.

Necessity: The preference relation induced by the utility functional (5) with $u(\mathrm{X})=\mathrm{R}$ satisfies (i), (ii) with the scale group $\mathrm{T}_u(\mathrm{X})$, and (iii) with (13), if $w(p^{(n-1)})\neq 0$, and with arbitrary $x\le x_n$ otherwise.

Sufficiency: By (12), each lottery has a certainty equivalent. From nondegeneracy of ($\mathrm{D}_{\mathrm{X}}^{(N)}$,$\succeq$), (i), and (ii) it follows that the certainty equivalent is unique. Therefore, Proposition 3 holds. In the case of its second alternative there is nothing to prove. In the case of its first alternative we must prove that the utility functional G defined in (19) has the form

$$G(\boldsymbol{y};\boldsymbol{p})=\sum_{i=1}^{n}\left[w(p^{(i)})-w(p^{(i-1)})\right]y_i\,,\ (\boldsymbol{y};\boldsymbol{p})\in\mathrm{D}_{\mathrm{R}}^{(N)}\,. \qquad (26)$$

The proof is by induction over n. (26) holds for $n=1$ (by definition) and $n=2$ (by Theorem 1). Let, by the inductive hypothesis, (26) holds for some $n\in\{2,...,N\}$. By (iii), for arbitrary $(\boldsymbol{y};\boldsymbol{p})$ and a probability $p\in[0,1-p^{(n-1)}]$ there exists $y\in\mathrm{R}$ such that

$$\sum_{i=1}^{n-1}\left[w(p^{(i)})-w(p^{(i-1)})\right]y_i+\left[1-w(p^{(n-1)})\right]y_n=G(\boldsymbol{y};\boldsymbol{p})=G\left(\boldsymbol{y},y_n;p_1,...,p_{n-1},p,1-p^{(n-1)}-p\right)=$$
$$=G\left(y,y_n;p^{(n-1)}+p,1-p^{(n-1)}-p\right)=w(p^{(n-1)}+p)y+\left[1-w(p^{(n-1)}+p)\right]y_n, \tag{27}$$

where the second equality holds due to coalescing.

Denote $p^{(n)}=p^{(n-1)}+p$. If $w(p^{(n)})\neq 0$ then from (27) it follows that

$$y=\sum_{i=1}^{n}\left[\frac{w(p^{(i)})}{w(p^{(n)})}-\frac{w(p^{(i-1)})}{w(p^{(n)})}\right]y_i\ ,$$

otherwise y can be chosen arbitrary such that $y\le y_n$.

Hence, for arbitrary $y_{n+1}\ge y_n$ we have

$$G\left(\boldsymbol{y},y_{n+1};p_1,...,p_{n-1},p,1-p^{(n)}\right)=G\left(y,y_{n+1};p^{(n)},1-p^{(n)}\right)=yw(p^{(n)})+\left[1-w(p^{(n)})\right]y_{n+1}=$$
$$=\sum_{i=1}^{n}\left[w(p^{(i)})-w(p^{(i-1)})\right]y_i+\left[1-w(p^{(n)})\right]y_{n+1}, \tag{28}$$

where the first equality holds due to independence of y and y_{n+1} (axiom (iii)).

(28) is exactly (26) for the lottery with $n+1$ outcomes. Therefore, (26) holds for all $n\le N$. ■

Proof of Theorem 3.

Necessity: Trivial.

Sufficiency: From (i) and (ii) it follows that there exists a continuous (with respect to outcomes) and idempotent utility functional that represents ($\mathrm{D}_{\mathrm{X}}^{(N)}$,$\succeq$) (Proposition 3). Hence, for any lottery $(\boldsymbol{x};\boldsymbol{p})\in\mathrm{D}_{\mathrm{X}}^{(N)}$ with $n\in\{3,...,N\}$ outcomes there exists $x\le x_n$ such that (11) holds. Moreover, by branch independence (iii), the value of x is independent of x_n. Thus, ($\mathrm{D}_{\mathrm{X}}^{(N)}$,$\succeq$) satisfies the conditions of Theorem 2. ■

To prove Theorem 4 we need the following variant of a well-known result on Jensen's functional equation on a restricted domain.

Proposition 5.

Let $\alpha,\beta\in(0,1)$ and let X be an open interval. If $f:\mathrm{X}^n\to\mathrm{R}$ is a nondecreasing solution of the equation

$$f\left(\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}\right)=\beta f(\boldsymbol{x})+(1-\beta)f(\boldsymbol{y})\ \text{on the set}\ \mathrm{X}_{\le}^{2n}=\{(\boldsymbol{x},\boldsymbol{y})\in\mathrm{X}^{2n}:\boldsymbol{x}\le\boldsymbol{y}\}, \tag{29}$$

then

$$f(\boldsymbol{x})=\sum_{i=1}^{n}w_ix_i+c \tag{30}$$

for some constants c and $w_i \geq 0$, $i = 1,...,n$. If f is not a constant function, then $\alpha = \beta$.

Proof.

Let $b \in \mathrm{R}$. Clearly, $f(\boldsymbol{x})$ is a solution of (29) if and only if the function $f(\boldsymbol{x}+b)$ is a solution of (29) on the set $(\mathrm{X}-b)^{2n}_{\leq}$. Hence, without loss of generality, we may assume that $0 \in \mathrm{X}$.

Substituting $\boldsymbol{x} = \boldsymbol{0}$ in (29), we get $f\big((1-\alpha)\boldsymbol{y}\big) = \beta f(\boldsymbol{0}) + (1-\beta) f(\boldsymbol{y})$, $\boldsymbol{y} \geq \boldsymbol{0}$. Substituting $\boldsymbol{y} = \boldsymbol{0}$ in (29), we obtain $f(\alpha\boldsymbol{x}) = \beta f(\boldsymbol{x}) + (1-\beta) f(\boldsymbol{0})$, $\boldsymbol{x} \leq \boldsymbol{0}$.

Therefore,

$$f\big(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\big) = \beta f(\boldsymbol{x}) + (1-\beta) f(\boldsymbol{y}) = f(\alpha\boldsymbol{x}) + f\big((1-\alpha)\boldsymbol{y}\big) - f(\boldsymbol{0}),\ \boldsymbol{x} \leq \boldsymbol{0},\ \boldsymbol{y} \geq \boldsymbol{0}.$$

Denote $g(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\boldsymbol{0})$, $\boldsymbol{x}' = \alpha\boldsymbol{x}$, $\boldsymbol{y}' = (1-\beta)\boldsymbol{y}$, then

$$g(\boldsymbol{x}' + \boldsymbol{y}') = g(\boldsymbol{x}') + g(\boldsymbol{y}'). \tag{31}$$

(31) is the Cauchy functional equation on the restricted domain $\{(\boldsymbol{x}', \boldsymbol{y}') \in \mathrm{X}^{2n} : \boldsymbol{x}' \leq \boldsymbol{0}, \boldsymbol{y}' \geq \boldsymbol{0}\}$. The general nondecreasing solution of (31) is given by (Radó and Baker, 1987) $g(\boldsymbol{z}) = \sum_{i=1}^{n} w_i z_i$ for some nonnegative constants $w_1,..., w_n$. Hence, (30) holds with $c = f(\boldsymbol{0})$.

If f is not a constant function then combining (29) and (30) we obtain $\alpha = \beta$. ■

Proof of Theorem 4.

Necessity: Trivial.

Sufficiency follows from Proposition 3. For its second alternative there is nothing to prove. Hence, we consider only the first alternative of Proposition 3. Obviously, a preference relation $(\mathrm{D}^{(N)}_{\mathrm{R}}, \succeq')$ induced by the utility functional (19) satisfies ordinal bisymmetry if and only if $(\mathrm{D}^{(N)}_{\mathrm{X}}, \succeq)$ does. By Theorem 1,

$$G\big(y_1, y_2; p, 1-p\big) = w(p) y_1 + \big(1 - w(p)\big) y_2.$$

If $w\big([0,1]\big) = \{0,1\}$ then the second alternative of Proposition 3 holds. We have to prove our statement in the case when there exists $p^* \in (0,1)$ such that $w(p^*) \in (0,1)$.

Ordinal bisymmetry (16) with $n = N$, $n' = 2$, and $\boldsymbol{p}' = (p^*, 1-p^*)$ reduces to Jensen's functional equation

$$w(p^*) G\big(\boldsymbol{y}^{(1)}; \boldsymbol{p}\big) + \big(1 - w(p^*)\big) G\big(\boldsymbol{y}^{(2)}; \boldsymbol{p}\big) = G\big(w(p^*)\boldsymbol{y}^{(1)} + \big(1 - w(p^*)\big)\boldsymbol{y}^{(2)}; \boldsymbol{p}\big)$$

on the domain $\mathrm{R}^{2N}_{\leq}$. By Proposition 5 and idempotence of G,

$$G\big(\boldsymbol{y}; \boldsymbol{p}\big) = \sum_{i=1}^{N} w_i^{(N)}(\boldsymbol{p}) y_i \tag{32}$$

for some functions $w_i^{(N)}: \mathrm{P}^N \to \mathrm{R}$, $i = 1,\ldots,N$ such that $\sum_{i=1}^{N} w_i^{(N)}(\boldsymbol{p}) \equiv 1$.

To determine functions $w_i^{(N)}$ consider the indifference relation

$$\left(\underbrace{y_1,\ldots,y_1}_{k}, \underbrace{y_2,\ldots,y_2}_{N-k}; \boldsymbol{p} \right) \sim' \left(y_1, y_2; p^{(k)}, 1-p^{(k)} \right), \tag{33}$$

following from coalescing (here $(\mathrm{D}_{\mathrm{R}}^{(N)}, \sim')$ is the symmetric part of $(\mathrm{D}_{\mathrm{R}}^{(N)}, \succeq')$). Take $y_1 = -1$, $y_2 = 0$ in (33), then using (32), we get $\sum_{i=1}^{k} w_i^{(N)}(\boldsymbol{p}) = w_1^{(2)}(p^{(k)})$.

Denote $w = w_1^{(2)}$, then

$$w_k^{(N)}(\boldsymbol{p}) = \sum_{i=1}^{k} w_i^{(N)}(\boldsymbol{p}) - \sum_{i=1}^{k-1} w_i^{(N)}(\boldsymbol{p}) = w(p^{(k)}) - w(p^{(k-1)}).^{2} \tag{34}$$

Because of stochastic dominance, the function w is nondecreasing and $0 \le w(p) \le 1$. By coalescing, $w(0) = 0$, $w(1) = 1$. Hence, for $n = N$ G has the form (26). The obtained result can be extended to $n < N$, by coalescing. ■

Proof of Theorem 5.

Necessity: Trivial.

Sufficiency: First we recall the notion of quasisum (due to Maksa, 2005). Let X_1 and X_2 be intervals and $\mathrm{X}_1 \times \mathrm{X}_2 \subseteq \mathrm{S} \subseteq \mathrm{R}^2$. A function $U: \mathrm{S} \to \mathrm{R}$ is a *quasisum* on $\mathrm{X}_1 \times \mathrm{X}_2$ if there exist continuous and strictly increasing functions $u_1: \mathrm{X}_1 \to \mathrm{R}$, $u_2: \mathrm{X}_2 \to \mathrm{R}$, $u_{12}^{-1}: u_1(\mathrm{X}_1) + u_2(\mathrm{X}_2) \to \mathrm{R}$ such that $U(x_1, x_2) = u_{12}^{-1}\left(u_1(x_1) + u_2(x_2)\right)$ for all $(x_1, x_2) \in \mathrm{X}_1 \times \mathrm{X}_2$. U is a *local quasisum* on $\mathrm{X}_1 \times \mathrm{X}_2$ if, for all $(x_1, x_2) \in \mathrm{X}_1 \times \mathrm{X}_2$, there exist open intervals X_1', X_2' such that $(x_1, x_2) \in \mathrm{X}_1' \times \mathrm{X}_2'$ and U is a quasisum on $(\mathrm{X}_1 \times \mathrm{X}_2) \cap (\mathrm{X}_1' \times \mathrm{X}_2')$.

Let conditions (i)–(iii) be satisfied. Then there exists an idempotent utility functional U that represents $(\mathrm{D}_{\mathrm{X}}^{(N)}, \succeq)$ and for each fixed $\boldsymbol{p} > 0$ $U(\cdot; \boldsymbol{p})$ is strictly increasing and continuous (Debreu, 1954). Ordinal bisymmetry (16) with $n = n' = 2$ and $\boldsymbol{p} = (p, 1-p)$, $\boldsymbol{p}' = (p', 1-p')$, $p, p' \in (0,1)$ reduces to the bisymmetry functional equation

$$\begin{aligned} &U\left(U\left(x_1^{(1)}, x_2^{(1)}; p, 1-p\right), U\left(x_1^{(2)}, x_2^{(2)}; p, 1-p\right); p', 1-p'\right) = \\ &= U\left(U\left(x_1^{(1)}, x_1^{(2)}; p', 1-p'\right), U\left(x_2^{(1)}, x_2^{(2)}; p', 1-p'\right); p, 1-p\right) \end{aligned} \tag{35}$$

on the restricted domain $\{x_{(i)}^{(j)} \in \mathrm{X}, i, j = 1,2 : x_{(i)}^{(j)} \le x_{(i')}^{(j')}, i \le i', j \le j', i, i', j, j' = 1,2\}$.[3]

For fixed $c_1 < c_2$ in X define $X_1 = (\inf X, c_1]$, $X_2 = [c_1, c_2]$, $X_3 = [c_2, \sup X)$, $X_{ii} = \{(x_1, x_2) \in X_i^2 : x_1 \le x_2\}$, $X_{ij} = X_i \times X_j$, $i < j$, $i, j \in \{1, 2, 3\}$. Clearly, $\{(x_1, x_2) \in X^2 : x_1 \le x_2\} = \bigcup_{i \le j} X_{ij}$.

The general continuous and strictly increasing solution of (35) with $p = p'$ on the rectangle domain $x_1^{(1)} \in X_1$, $x_1^{(2)}, x_2^{(1)} \in X_2$, $x_2^{(2)} \in X_3$ is known (Maksa, 1999, Theorem 1). In particular, for a fixed p $U(\cdot,\cdot; p, 1-p)$ is a quasisum on X_{12} and on X_{23}.

From (35) with $p = p'$ on the domain $(x_1^{(1)}, x_1^{(2)}) \in X_{11}$, $(x_2^{(1)}, x_2^{(2)}) \in X_{22}$, we get

$$
\begin{aligned}
&U\left(u_{12}^{-1}\left(u_1(x_1^{(1)}) + u_2(x_2^{(1)})\right), u_{12}^{-1}\left(u_1(x_1^{(2)}) + u_2(x_2^{(2)})\right)\right) = \\
&= u_{12}^{-1}\left(u_1 \circ U\left(x_1^{(1)}, x_1^{(2)}\right) + u_2 \circ U\left(x_2^{(1)}, x_2^{(2)}\right)\right)
\end{aligned}
\tag{36}
$$

for some continuous and strictly increasing functions $u_1 : X_1 \to R$, $u_2 : X_2 \to R$, $u_{12}^{-1} : u_1(X_1) + u_2(X_2) \to R$ (here we use the fact that U is a quasisum on X_{12}). In (36) and hereafter, for convenience, we drop the subscripts p and $1-p$ from U.

By definition, put $u_i(X_{ii}) = \{(u_i(x_1), u_i(x_2)) : (x_1, x_2) \in X_{ii}\}$, $y_i^{(j)} = u_i(x_i^{(j)})$, $i, j = 1, 2$, $F(y_1, y_2) = u_{12} \circ U\left(u_{12}^{-1}(y_1), u_{12}^{-1}(y_2)\right)$, $G(y_1, y_2) = u_1 \circ U\left(u_1^{-1}(y_1), u_1^{-1}(y_2)\right)$, and $H(y_1, y_2) = u_2 \circ U\left(u_2^{-1}(y_1), u_2^{-1}(y_2)\right)$, then (36) reduces to

$$
F\left(y_1^{(1)} + y_2^{(1)}, y_1^{(2)} + y_2^{(2)}\right) = G\left(y_1^{(1)}, y_1^{(2)}\right) + H\left(y_2^{(1)}, y_2^{(2)}\right).
\tag{37}
$$

(37) is Pexider's functional equation on the restricted domain $(y_i^{(1)}, y_i^{(2)}) \in u_i(X_{ii})$, $i = 1, 2$. Its general idempotent strictly increasing solution is given by (Radó and Baker, 1987)

$$G(y_1, y_2) = wy_1 + (1-w)y_2, \ (y_1, y_2) \in u_1(X_{11}),$$

$$H(y_1, y_2) = wy_1 + (1-w)y_2, \ (y_1, y_2) \in u_2(X_{22})$$

for some $w \in (0,1)$.

Hence,

$$U(x_1, x_2) = u_1^{-1}\left(wu_1(x_1) + (1-w)u_1(x_2)\right), \ (x_1, x_2) \in X_{11},$$

$$U(x_1, x_2) = u_2^{-1}\left(wu_2(x_1) + (1-w)u_2(x_2)\right), \ (x_1, x_2) \in X_{22}.$$

Define the continuous and strictly increasing function $u' : X_1 \cup X_2 \to R$ by

$$
u'(x) = \begin{cases} \dfrac{u_1(x)}{w} + \dfrac{u_2(c_1)}{1-w} & \text{if } x \in X_1 \\ \dfrac{u_2(x)}{1-w} + \dfrac{u_1(c_1)}{w} & \text{if } x \in X_2 \end{cases}.
$$

Then

$$U(x_1, x_2) = u_{12}^{-1}\left(wu'(x_1) + (1-w)u'(x_2) - \frac{wu_2(c_1)}{1-w} - \frac{(1-w)u_1(c_1)}{w} \right),\ (x_1, x_2) \in \mathrm{X}_{12} \tag{38}$$

and

$$U(x_1, x_2) = u'^{-1}(wu'(x_1) + (1-w)u'(x_2)) \tag{39}$$

whenever either $(x_1, x_2) \in \mathrm{X}_{11}$ or $(x_1, x_2) \in \mathrm{X}_{22}$. From (38) with $x_1 = c_1$ and (39), we get

$$u_{12}^{-1}\left(y - \frac{wu_2(c_1)}{1-w} - \frac{(1-w)u_1(c_1)}{w} \right) = u'^{-1}(y),\ y \in [u'(c_1), wu'(c_1) + (1-w)u'(c_2)]. \tag{40}$$

Using (38) with $x_2 = c_1$ and (39), we obtain

$$u_{12}^{-1}\left(y - \frac{wu_2(c_1)}{1-w} - \frac{(1-w)u_1(c_1)}{w} \right) = u'^{-1}(y),\ y \in (\inf u'(\mathrm{X})w + (1-w)u'(c_1), u'(c_1)]. \tag{41}$$

Combining (38)–(41), we get that (39) holds for $(x_1, x_2) \in \mathrm{X}_{11} \cup \mathrm{X}_{12} \cup \mathrm{X}_{22}$.

From (35) with $p = p'$ on the domain $(x_1^{(1)}, x_1^{(2)}) \in \mathrm{X}_{22}$, $(x_2^{(1)}, x_2^{(2)}) \in \mathrm{X}_{33}$, in the same way, we obtain

$$U(x_1, x_2) = u''^{-1}(w'u''(x_1) + (1-w')u''(x_2)),\ (x_1, x_2) \in \mathrm{X}_{22} \cup \mathrm{X}_{23} \cup \mathrm{X}_{33} \tag{42}$$

for a continuous and strictly increasing function $u'' : \mathrm{X}_2 \cup \mathrm{X}_3 \to \mathrm{R}$.

Comparing (39) and (42) on the set $(x_1, x_2) \in \mathrm{X}_{22}$, we get

$$u'^{-1}(wu'(x_1) + (1-w)u'(x_2)) = u''^{-1}(w'u''(x_1) + (1-w')u''(x_2)),\ (x_1, x_2) \in \mathrm{X}_{22}.$$

This is Jensen's functional equation with respect to the function $u'' \circ u'^{-1}$. Hence, by Proposition 5, $u'' = au' + b$, $a > 0$ on X_2 and $w = w'$. Since the functions u' and u'' are defined up to an affine transformation, without loss of generality, we may assume that $u'' = u'$ on X_2 and define a continuous and strictly increasing function

$$u(x) = \begin{cases} u'(x) & \text{if } x \in \mathrm{X}_1 \cup \mathrm{X}_2 \\ u''(x) & \text{if } x \in \mathrm{X}_3 \end{cases}.$$

Hence,

$$U(x_1, x_2) = u^{-1}(wu(x_1) + (1-w)u(x_2)) \tag{43}$$

whenever $(x_1, x_2) \in \mathrm{X}_{11} \cup \mathrm{X}_{12} \cup \mathrm{X}_{22} \cup \mathrm{X}_{23} \cup \mathrm{X}_{33}$.

To determine U on X_{13} we use the fact that U is a quasisum on X_{13}. Indeed, since $\mathrm{X}_1 \cap \mathrm{X}_3 = \varnothing$ and X is open, given $(x_1, x_2) \in \mathrm{X}_{13}$ there exist c_1', c_2' in X such that $x_1 < c_1' < x_2 < c_2'$. From (35) with $p = p'$ on the domain $x_1^{(1)} \in (\inf \mathrm{X}, c_1')$, $(x_1^{(2)}, x_2^{(1)}) \in (c_1', c_2')^2$, $x_2^{(2)} \in (c_2', \sup \mathrm{X})$, we get (Maksa, 1999, Theorem 1) that U is a quasisum on $(\inf \mathrm{X}, c_1') \times (c_1', c_2')$. Hence, U is a local quasisum on X_{13}. But a local quasisum on a rectangle is a quasisum on it

(Maksa, 2005, Theorem 1). Therefore, there exist continuous and strictly increasing functions u_{13}, u_1, u_3 such that $U(x_1, x_2) = u_{13}^{-1}(u_1(x_1) + u_3(x_2))$ on X_{13}.

From (35) with $p = p'$ on the domain $(x_1^{(1)}, x_1^{(2)}) \in X_{11}$, $(x_2^{(1)}, x_2^{(2)}) \in X_{33}$, in the same way, we obtain that (43) holds also for X_{13}. Hence,

$$U(x_1, x_2; p, 1-p) = u_p^{-1}(w(p)u_p(x_1) + (1-w(p))u_p(x_2))$$

for some function $u_p : X \to R$, depending on p.

To determine whether u_p actually depends on p consider (35) for $p \neq p'$:

$$\begin{aligned} &U\left(u_p^{-1}\left(w(p)u_p(x_1^{(1)}) + (1-w(p))u_p(x_2^{(1)})\right), u_p^{-1}\left(w(p)u_p(x_1^{(2)}) + (1-w(p))u_p(x_2^{(2)})\right); p', 1-p'\right) = \\ &= u_p^{-1}\left(w(p)u_p \circ U\left(x_1^{(1)}, x_1^{(2)}; p', 1-p'\right) + (1-w(p))u_p \circ U\left(x_2^{(1)}, x_2^{(2)}; p', 1-p'\right)\right). \end{aligned} \tag{44}$$

(44) is Jensen's functional equation with respect to the function $(y_1, y_2) \mapsto u_p \circ U\left(u_p^{-1}(y_1), u_p^{-1}(y_2); p', 1-p'\right)$. By Proposition 5 and idempotence of U, there exists a function $w' : [0,1]^2 \to [0,1]$ such that

$$\begin{aligned} &u_{p'}^{-1}\left(w(p')u_{p'}(x_1) + (1-w(p'))u_{p'}(x_2)\right) = U(x_1, x_2; p', 1-p') = \\ &= u_p^{-1}\left(w'(p, p')u_p(x_1) + (1-w'(p, p'))u_p(x_2)\right). \end{aligned}$$

This is again Jensen's equation with respect to the function $u_p \circ u_{p'}^{-1}$. Hence, by Proposition 5, $u_p = u_{p'}$ (up to an affine transformation) and $w'(p, p') = w(p')$.

Therefore,

$$U(x_1, x_2; p, 1-p) = u^{-1}(w(p)u(x_1) + (1-w(p))u(x_2)),\ x_1 \le x_2,\ (x_1, x_2) \in X^2,\ p \in [0,1]. \tag{45}$$

w is strictly increasing, by (i).

From the proof of Theorem 4 we know that (45) and ordinal bisymmetry (16) imply RDU. ■

Proof of Theorem 7.

Necessity: a preference relation induced by (17) with $u(X) = R$ satisfies (i), (ii), (iii), and (iv) with $g(x, x') = u^{-1}(u(x) + u(x'))$.

Sufficiency: Let U be an idempotent utility functional that represents $(D_X'^{(\infty)}, \succeq)$ ((iii) and Proposition 1). Taking into account Theorem 1, we obtain that there exist an increasing bijection $u : X \to R$ and a transformation function w such that

$$U(x_1, x_2; p, 1-p) = u^{-1}(u(x_1)w(p) + u(x_2)(1-w(p))),\ (x_1, x_2; p, 1-p) \in D_X^{(2)}.$$

w is continuous, by (iii). Therefore, there exists $p^* \in (0,1)$ such that $w(p^*) \in (0,1)$.

Let $(\mathrm{D}'^{(\infty)}_{\mathrm{R}}, \succeq')$ be the preference relation on the set $\mathrm{D}'^{(\infty)}_{\mathrm{R}}$ defined by (8) and let G be the idempotent utility functional (19). $\oplus$ induces the following concatenation operation $\oplus'$ on $\mathrm{D}'^{(\infty)}_{\mathrm{R}}$:

$$(\boldsymbol{y};\boldsymbol{p})\oplus'(\boldsymbol{y}';\boldsymbol{p}') = \left(f(y_1,y'_1), f(y_1,y'_2),\ldots, f(y_n,y'_{n'}); p_1p'_1, p_1p'_2,\ldots, p_np'_{n'}\right),$$

where $f(y,y') = u\circ g\left(u^{-1}(y),u^{-1}(y')\right)$. Obviously, $(\mathrm{D}'^{(\infty)}_{\mathrm{R}}, \succeq')$ satisfies (i), (ii), (iii), and (iv) (with $\oplus'$ instead of $\oplus$) if and only if $(\mathrm{D}'^{(\infty)}_{\mathrm{X}}, \succeq)$ does.

Given $y_1 \le y_2$ let δ_y be a certainty equivalent of the lottery $\left(y_1,y_2;p^*,1-p^*\right)\in \mathrm{D}^{(2)}_{\mathrm{R}}$. By (iv), for any $y'\in \mathrm{R}$

$$\begin{aligned} &f(y_1,y')w(p^*) + f(y_2,y')\left(1-w(p^*)\right) = G\left(\left(y_1,y_2;p\right)\oplus'\delta_{y'}\right) = G\left(\delta_y \oplus' \delta_{y'}\right) = \\ &= G\left(\delta_{f(y,y')}\right) = f(y,y') = f\left(y_1w(p^*) + y_2\left(1-w(p^*)\right), y'\right). \end{aligned} \tag{46}$$

For a fixed y' (46) is Jensen's functional equation with respect to the function $f(\cdot,y')$. Its general nondecreasing solution is given by (Proposition 5)

$$f(y,y') = a(y')y + b(y') \tag{47}$$

for some functions a and b.

By symmetry of f, it admits the dual representation

$$f(y,y') = a(y)y' + b(y). \tag{48}$$

Combining (47) and (48), we obtain the bilinear functional equation. Its general solution is given by (Aczél, 1966, p. 161)

$$f(y,y') = a(y+y') + b + cyy'$$

for some constants a, b, and c (compare with Luce and Fishburn, 1995, p. 8, equation (11)). Since f is nonconstant and nondecreasing,

$$f(y,y') = a(y+y') + b \tag{49}$$

for some constants $a > 0$ and b.

From (49) and (iv) it follows that

$$\begin{aligned} &G\left(a(y_1+y'_1)+b, a(y_1+y'_2)+b,\ldots, a(y_n+y'_{n'})+b; p_1p'_1, p_1p'_2,\ldots, p_np'_{n'}\right) = \\ &= G\left(\left(\boldsymbol{y};\boldsymbol{p}\right)\oplus'\left(\boldsymbol{y}';\boldsymbol{p}'\right)\right) = a\left(G\left(\boldsymbol{y};\boldsymbol{p}\right) + G\left(\boldsymbol{y}';\boldsymbol{p}'\right)\right) + b. \end{aligned} \tag{50}$$

Combining (9) and (50), we obtain an additive representation:

$$G\left(y_1+y'_1, y_1+y'_2,\ldots, y_n+y'_{n'}; p_1p'_1, p_1p'_2,\ldots, p_np'_{n'}\right) = G\left(\boldsymbol{y};\boldsymbol{p}\right) + G\left(\boldsymbol{y}';\boldsymbol{p}'\right). \tag{51}$$

Given $\left(\boldsymbol{y};\boldsymbol{p}\right)\in \mathrm{D}'^{(\infty)}_{\mathrm{R}}$ let y be the mathematical expectation $y = \sum_i p_i y_i$. By the definition of $\mathrm{D}^{(\infty)}_{\mathrm{R}}$ (see section 2), y is well defined.

Define inductively

$$\left(\boldsymbol{y}^{(k+1)};\boldsymbol{p}^{(k+1)}\right) = \left(y_1^{(k)} + y_1, y_1^{(k)} + y_2,\ldots, y_{n^k}^{(k)} + y_n; p_1^{(k)}p_1, p_1^{(k)}p_2,\ldots, p_{n^k}^{(k)}p_n\right),$$

$\left(\boldsymbol{y}^{(1)};\boldsymbol{p}^{(1)}\right)=(\boldsymbol{y};\boldsymbol{p}),\ k=1,2,...$

From (9) and (51), we get

$G(\boldsymbol{y};\boldsymbol{p})=\sum_{j=1}^{k}G(\boldsymbol{y};\boldsymbol{p})/k=\sum_{j=1}^{k}G(\boldsymbol{y}/k;\boldsymbol{p})=G\left(\boldsymbol{y}^{(k)}/k;\boldsymbol{p}^{(k)}\right)$ for any natural k.

By the law of large numbers $\left(\boldsymbol{y}^{(k)}/k;\boldsymbol{p}^{(k)}\right)$ converges weakly to δ_y as $k\to\infty$. Hence, by (iii),

$G(\boldsymbol{y};\boldsymbol{p})=G\left(\delta_y\right)=\sum_i p_i y_i$.

Applying the reverse transformation, we obtain EU representation for ($\mathrm{D}'^{(\infty)}_{\mathrm{X}}$,$\succeq$). ■

Notes

1. If the inverse inequality holds, then consider T^{-1} instead of T.
2. The proof that (32) implies (34) also follows from a general result of Luce and Marley (2005, Theorem 11).
3. Recently professor Imre Kocsis (2009) (personal communications) has considered a generalized version of equation (35).

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Address for correspondence: Mikhail V. Sokolov. St. Petersburg Institute for Economics and Mathematics RAS, 1 Chaikovskogo st., St. Petersburg, 191187, Russia; European University at St. Petersburg. 3 Gagarinskaya st., St. Petersburg, 191187, Russia. E-mail: sos-homepage@yandex.ru.